# THE MAGIC OF PASCAL'S TRIANGLE

## PASCAL'S TRIANGLE

This represents a way to write down the "early" binomial coefficients $\binom{n}{r}$ easily.

• Each row begins and ends with "1". (We have a "tent" of "1"s.)
• Every other entry equals the sum of the two entries immediately above it.

Here it is:

| | |
|---|---|
| 1 | Row 0: Contains $\binom{0}{0}$ |
| 1 1 | Row 1: Contains $\binom{1}{0},\binom{1}{1}$ |
| 1 2 1 | Row 2: Contains $\binom{2}{0},\binom{2}{1},\binom{2}{2}$ |
| 1 3 3 1 | Row 3: Contains $\binom{3}{0},\binom{3}{1},\binom{3}{2},\binom{3}{3}$ |
| 1 4 6 4 1 | Row 4: Contains $\binom{4}{0},\binom{4}{1},\binom{4}{2},\binom{4}{3},\binom{4}{4}$ |

etc. - The "histograms" of the rows approach a bell-shaped "normal" distribution!

We can observe some basic properties of binomial coefficients:

Symmetry about the center: $\binom{n}{r}=\binom{n}{n-r}$

(The process of choosing $r$ winners is equivalent to the process of choosing $n$-$r$ losers.)

The "tent" of "1"s: $\binom{n}{0}=\binom{n}{n}=1$

The "inner tent" of natural numbers: $\binom{n}{1}=\binom{n}{n-1}=n$

(There are $n$ ways to get one winner and $n$-1 losers from a group of $n$ people.)

**THE PLINKO / PACHINKO APPROACH TO PASCAL'S TRIANGLE**

In the game of Plinko on the CBS game show "The Price is Right", contestants won money by standing over a board full of pins and dropping chips. The chips then fell into bins at the bottom of the board; each bin was labeled with a dollar amount that was added to the contestant's winnings.

Now imagine a pinboard shaped like Pascal's triangle:

Player

1 ← After pin is hit, the chip can move left or right.

1 1 ← After a pin is hit, the chip can move left or right.

1 2 1 ← After a pin is hit, the chip can move left or right.

1 **3** 3 1 ← After a pin is hit, the chip can move left or right.

1 4 6 4 1 ← After a pin is hit, the chip can move left or right.

It turns out that the number of ways to hit a pin is equal to the number on the pin!

Example

There are 3 ways to hit the boldfaced pin labeled "3" above.

One way: LLR

**1**

← Left (L) - first "drop"

**1** 1

← Left (L) - second "drop"

**1** 2 1

← Right (R) - third "drop"

1 **3** 3 1

1 4 6 4 1

A second way: LRL

**1**

← Left (L)

**1** 1

← Right (R)

1 **2** 1

← Left (L)

1 **3** 3 1

1 4 6 4 1

<u>A third way</u>:  RLL

**1**

← Right (R)

1 **1**

← Left (L)

1 **2** 1

← Left (L)

1 **3** 3 1

1 4 6 4 1

The only way to reach the "**3**" is if there are exactly one "R" and two "L"s in any order. How many ways are there to choose exactly one of the three drops to be an "R"? $\binom{3}{1}$, which the 3 represents!

Likewise, there are 6 ways to hit the "6" pin, because out of four drops, we need exactly two of them to be "R"s. There are $\binom{4}{2} = 6$ ways to do this.

Remember the construction of Pascal's triangle:

- Each row begins and ends with "1". (We have a "tent" of "1"s.)
- Every other entry equals <u>the sum of the two entries immediately above it</u>.

How does the Plinko model exhibit this?

<u>Example</u>

1

1 1

1 2 1

**1** **3** 3 1

1 **4** 6 4 1

There is 1 way to hit the "**1**" (all "L"s).
Then, you can move right to hit the "**4**".

There are 3 ways to hit the "**3**".
Then, you can move left to hit the "**4**".

So, there are 4 total ways to hit the "**4**".

<u>Note</u>: If you go to a science museum that has a "Plinko-type" board with numerous balls being dropped from the top-center, the bottom of the board should look like a bell curve!

## EXPANDING POWERS OF BINOMIALS: THE BINOMIAL THEOREM (MATH 96)

Remember that Row $n$ of Pascal's Triangle provides the coefficients for the expansion of $(a+b)^n$:

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{n}b^n$$

Example

$$(a+b)^3 = 1a^3 + 3a^2b + 3ab^2 + 1b^3$$

Here's why this expansion of $(a+b)^3$ is correct:

You can start by writing down 8 terms, each corresponding to a sequence of choices.

$$(a+b)^3 = \underbrace{(a+b)}_{\substack{\text{Choose} \\ \text{"}a\text{" or "}b\text{"}}} \underbrace{(a+b)}_{\substack{\text{Choose} \\ \text{"}a\text{" or "}b\text{"}}} \underbrace{(a+b)}_{\substack{\text{Choose} \\ \text{"}a\text{" or "}b\text{"}}}$$

$$(a+b)^3 = \underbrace{aaa}_{a^3} \quad \leftarrow \begin{cases} \binom{3}{0} = 1 \text{ way to never} \\ \text{choose "}b\text{"} \end{cases}$$

$$+ \underbrace{aab + aba + baa}_{3a^2b} \quad \leftarrow \begin{cases} \binom{3}{1} = 3 \text{ ways to choose "}b\text{"} \\ \text{exactly once among the 3 factors} \end{cases}$$

$$+ \underbrace{abb + bab + bba}_{3ab^2} \quad \leftarrow \begin{cases} \binom{3}{2} = 3 \text{ ways to choose "}b\text{"} \\ \text{exactly 2 times among the 3 factors} \end{cases}$$

$$+ \underbrace{bbb}_{b^3} \quad \leftarrow \begin{cases} \binom{3}{3} = 1 \text{ way to always} \\ \text{choose "}b\text{"} \end{cases}$$

$$= a^3 + 3a^2b + 3ab^2 + b^3$$

Notice that choosing the "$a$" or the "$b$" from each factor is analogous to dropping "left" or "right" at each step in our Plinko model!