SOLUTIONS TO THE FINAL

MATH 121 – FALL 2003 – KUNIYUKI
126 POINTS TOTAL, BUT 120 POINTS = 100%

1) Approximate the area under the graph of $f(x) = \dfrac{1}{x}$ from $a = 2$ to $b = 14$ by finding a Left Riemann Sum using 4 rectangles of the same width. Round off to four decimal places whenever you need to round off. (14 points)

Step 1: Find Δx, the width of each rectangle.

$$\Delta x = \frac{b-a}{n} = \frac{14-2}{4} = \frac{12}{4} = 3$$

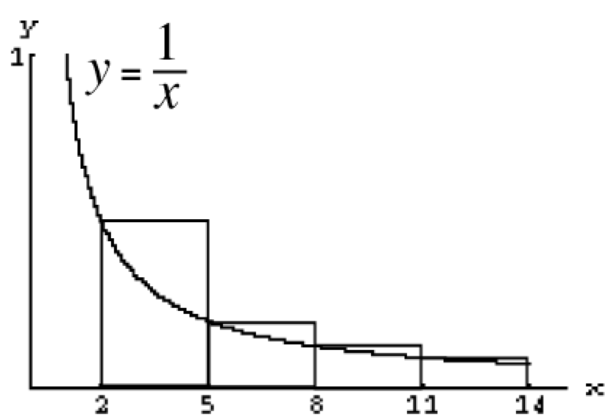
Step 2: Find the left breakpoints.

$$a = x_1 = 2 \xrightarrow{+3} x_2 = 5 \xrightarrow{+3} x_3 = 8 \xrightarrow{+3} x_4 = 11$$

Step 3: Find the Left Riemann Sum.

$$\begin{aligned} & f(2)\cdot\Delta x + f(5)\cdot\Delta x + f(8)\cdot\Delta x + f(11)\cdot\Delta x \\ &= \left(\frac{1}{2}\right)(3) + \left(\frac{1}{5}\right)(3) + \left(\frac{1}{8}\right)(3) + \left(\frac{1}{11}\right)(3) \\ &\approx 1.5 + 0.6 + 0.375 + 0.2727 \\ &= \mathbf{2.7477 \ \ square\ units} \end{aligned}$$

Note: The exact value is closer to 1.9459.



2) Flubber is consumed at the rate of $f(t)=3t^2$ tons per year, where t is the number of years since January 1, 2000. How much Flubber is consumed from January 1, 2002 to January 1, 2005? (9 points)

$$\int_2^5 3t^2\,dt = \left[t^3\right]_2^5$$
$$= \left[(5)^3\right]-\left[(2)^3\right]$$
$$= 125-8$$
$$= \mathbf{117\ tons}$$

3) Find the integrals. Simplify wherever possible. (50 points total)

a) $\int_1^5 \underbrace{\left(x^3-7x^{-2}\right)}_{\text{cont. on } [1,5]} dx$ (8 points)

You may write your final answer as a decimal.

$$= \left[\frac{x^4}{4}-7\left(\frac{x^{-1}}{-1}\right)\right]_1^5$$
$$= \left[\frac{x^4}{4}+\frac{7}{x}\right]_1^5$$
$$= \left[\frac{(5)^4}{4}+\frac{7}{(5)}\right]-\left[\frac{(1)^4}{4}+\frac{7}{(1)}\right]$$
$$= 157.65-7.25$$
$$= \mathbf{150.4}\ \text{or}\ \mathbf{\frac{752}{5}}\ \text{or}\ \mathbf{150\frac{2}{5}}$$

b) $\int_0^5 \underbrace{8e^{2x}}_{\substack{\text{cont.} \\ \text{on } [0,5]}}\, dx$ (7 points)

$$= \left[8\left(\frac{e^{2x}}{2}\right)\right]_0^5$$
$$= \left[4e^{2x}\right]_0^5$$
$$= \left[4e^{2(5)}\right]-\left[4e^{2(0)}\right]$$
$$= 4e^{10}-4\underbrace{e^0}_{=1}$$
$$= \mathbf{4e^{10}-4}$$

c) $\int x^2\left(x^3+9\right)^7 dx$ (9 points)

$$u = x^3 + 9$$
$$du = 3x^2\,dx$$

$$= \frac{1}{3}\int 3x^2\left(x^3+9\right)^7 dx$$
$$= \frac{1}{3}\int u^7\,du$$
$$= \frac{1}{3}\left(\frac{u^8}{8}\right) + C$$
$$= \frac{1}{24}u^8 + C$$
$$= \boldsymbol{\frac{1}{24}\left(x^3+9\right)^8 + C}$$

d) $\int \frac{5x+2}{5x^2+4x}\,dx$ (8 points)

$$u = 5x^2 + 4x$$
$$du = (10x+4)\,dx$$
$$= 2(5x+2)\,dx$$

$$= \frac{1}{2}\int \frac{2(5x+2)}{5x^2+4x}\,dx$$
$$= \frac{1}{2}\int \frac{du}{u}$$
$$= \frac{1}{2}\ln|u| + C$$
$$= \boldsymbol{\frac{1}{2}\ln\left|5x^2+4x\right| + C}$$

e) $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$ (8 points)

$$u = \sqrt{x} = x^{1/2}$$
$$du = \frac{1}{2}x^{-1/2}\,dx = \frac{1}{2\sqrt{x}}\,dx$$

$$= 2\int \frac{e^{\sqrt{x}}}{2\sqrt{x}}\,dx$$
$$= 2\int e^u\,du$$
$$= 2e^u + C$$
$$= \boldsymbol{2e^{\sqrt{x}} + C}$$

f) $\int_2^5 \frac{dx}{3-4x}$ (10 points)

(The integrand is continuous on [2,5].) New limits:

$u = 3-4x$ $x = 2 \Rightarrow u = 3-4(2) = -5$

$du = -4\,dx$ $x = 5 \Rightarrow u = 3-4(5) = -17$

$$= -\frac{1}{4}\int_{x=2}^{x=5} \frac{-4\,dx}{3-4x}$$

$$= -\frac{1}{4}\int_{u=-5}^{u=-17} \frac{du}{u}$$

$$= -\frac{1}{4}\left[\ln|u|\right]_{-5}^{-17}$$

$$= -\frac{1}{4}\left(\ln|-17| - \ln|-5|\right)$$

$$= -\mathbf{\frac{1}{4}}\left(\mathbf{\ln 17 - \ln 5}\right)$$

4) Find the average value of $f(x) = 5x^2$ on the interval $[1,4]$. (10 points)

$$f_{av} = \frac{\int_a^b f(x)\,dx}{b-a}$$

$$= \frac{\int_1^4 5x^2\,dx}{4-1}$$

$$= \frac{\left[5\left(\frac{x^3}{3}\right)\right]_1^4}{3}$$

$$= \frac{\left[\frac{5x^3}{3}\right]_1^4}{3}$$

$$= \frac{\left[\frac{5(4)^3}{3} - \frac{5(1)^3}{3}\right]}{3}$$

$$= \frac{\frac{320}{3} - \frac{5}{3}}{3}$$

$$= \frac{\frac{315}{3}}{3}$$

$$= \frac{105}{3}$$

$$= \mathbf{35}$$

5) Find the area bounded by the graphs of $y = x^2 + 2$ and $y = 5 - 2x$. (16 points)

We need to set up: $\int_a^b [(\text{top}) - (\text{bottom})]\,dx$.

Step 1: Where are the intersection points?

Solve $\begin{cases} y = x^2 + 2 & \text{(parabola)} \\ y = 5 - 2x & \text{(line)} \end{cases}$ for x, at least.

$$x^2 + 2 = 5 - 2x$$
$$x^2 + 2x - 3 = 0$$
$$(x+3)(x-1) = 0$$

The intersection points are at $x = -3$ and $x = 1$.

Step 2: Who's on top?

Test $x = 0$, since 0 is between -3 and 1.

$$y = x^2 + 2 \xrightarrow{x=0} y = 2$$
$$y = 5 - 2x \xrightarrow{x=0} y = 5$$

The graph of the second equation is on top.

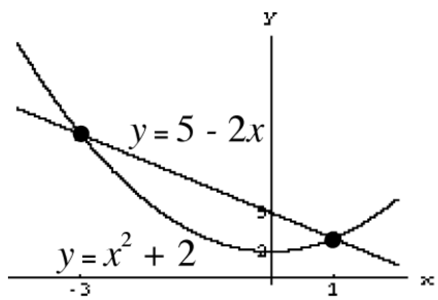
Also, observe that the bounded region has to have the line on top and the [upward-opening] parabola on the bottom.

Step 3: Set up the definite integral.

$$\int_{-3}^{1} \left[(5 - 2x) - (x^2 + 2)\right] dx$$
$$= \int_{-3}^{1} \left[5 - 2x - x^2 - 2\right] dx$$
$$= \int_{-3}^{1} \left[-x^2 - 2x + 3\right] dx$$
$$= \left[-\frac{x^3}{3} - x^2 + 3x\right]_{-3}^{1}$$
$$= \left[-\frac{(1)^3}{3} - (1)^2 + 3(1)\right] - \left[-\frac{(-3)^3}{3} - (-3)^2 + 3(-3)\right]$$
$$= \left[-\frac{1}{3} - 1 + 3\right] - [9 - 9 - 9]$$
$$= \left[2 - \frac{1}{3}\right] - [-9]$$

$$= \left[\frac{5}{3}\right] - [-9]$$
$$= \frac{5}{3} + 9$$
$$= \frac{5}{3} + \frac{27}{3}$$
$$= \mathbf{\frac{32}{3} \text{ or } 10\frac{2}{3}}$$

Note: Here is a graph of the region of interest:



6) Find the domain of $f(x,y) = \frac{\sqrt{x}}{y}$. (3 points)

$$\boldsymbol{\{(x,y) \mid x \geq 0, y \neq 0\}}$$

7) Let $f(x,y) = \ln\left(3x + y^3\right) + xy^2$. (15 points total)

a) Find $f_x(x,y)$.

$$f(x,y) = \ln\left(3x + \underbrace{y^3}_{\#}\right) + x\underbrace{y^2}_{\#}$$
$$f_x(x,y) = \frac{1}{3x + y^3} \cdot D_x\left(3x + \underbrace{y^3}_{\#}\right) + y^2$$
$$= \frac{1}{3x + y^3} \cdot 3 + y^2$$
$$= \mathbf{\frac{3}{3x + y^3} + y^2}$$

b) Find $f_y(x,y)$.

$$f(x,y) = \ln\Big(\underbrace{3x}_{\#} + y^3\Big) + \underbrace{x}_{\#}y^2$$

$$f_y(x,y) = \frac{1}{3x+y^3} \cdot D_y\Big(\underbrace{3x}_{\#} + y^3\Big) + x \cdot 2y$$

$$= \frac{1}{3x+y^3} \cdot 3y^2 + 2xy$$

$$= \mathbf{\frac{3y^2}{3x+y^3} + 2xy}$$

c) Find $f_y(1,2)$.

$$f_y(1,2) = \frac{3(2)^2}{3(1)+(2)^3} + 2(1)(2)$$

$$= \frac{12}{11} + 4$$

$$= \frac{12}{11} + \frac{44}{11}$$

$$= \mathbf{\frac{56}{11} \text{ or } 5\frac{1}{11}}$$

8) Let $f(x,y,z) = yz^4 - xe^y$. (9 points total)

a) Find $f(-2,3,1)$.

$$f(-2,3,1) = (3)(1)^4 - (-2)e^{(3)}$$

$$= \mathbf{3 + 2e^3}$$

b) Find $f_y(x,y,z)$.

$$f(x,y,z) = y\underbrace{z^4}_{\#} - \underbrace{x}_{\#}e^y$$

$$f_y(x,y,z) = \mathbf{z^4 - xe^y}$$

c) Find $f_{yz}(x,y,z)$.

$$f_y(x,y,z) = z^4 - \underbrace{xe^y}_{\#}$$

$$f_{yz}(x,y,z) = \mathbf{4z^3}$$