QUIZ #3 (SECTIONS 3.1, 3.2, 3.3, 3.6) SOLUTIONS

MATH 121 – FALL 2003 – KUNIYUKI
105 POINTS TOTAL, BUT 100 POINTS = 100%

1) Sketch the graph of $f(x) = x^4 - 4x^3 + 3$. You must:

- Find and label all points at critical numbers and inflection points (if any).
- Classify all points at critical numbers as relative maximum points, relative minimum points, or neither.
- Find the y-intercept.
- Have your graph correctly show where f is increasing / decreasing, and where f is concave up / concave down.
- Show <u>all</u> steps, as we have done in class.

(30 points)

Step 1: Domain = **R**.

Step 2: Find f' and critical numbers (CNs).

$$\begin{aligned} f(x) &= x^4 - 4x^3 + 3 \\ f'(x) &= 4x^3 - 12x^2 \\ &= 4x^2(x-3) \end{aligned}$$

This is never DNE. It equals 0 at $x = 0$ and $x = 3$, which are in the domain of f. The CNs are 0 and 3.

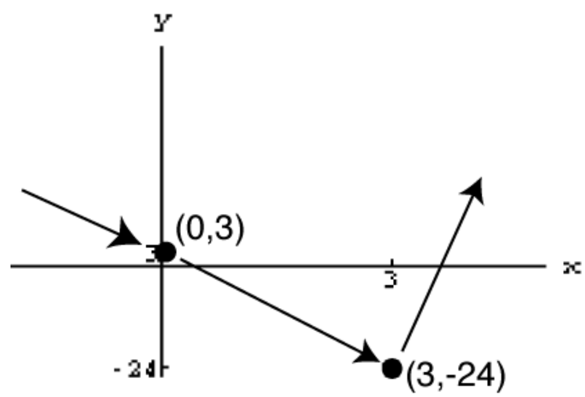
Step 3: Do a sign diagram for f' and classify the points at the CNs.

	Test $x = -1$	**0**	Test $x = 1$	**3**	Test $x = 4$
f' sign	–		–		+
f	↘		↘		↗
Classify Points at CNs (Using 1st DT.)		**Neither** R.Max. nor R.Min. Pt.		**R.Min. Pt.**	
Plug into $f(x)$ to get y		$(0, f(0))$ **$(0,3)$**		$(3, f(3))$ **$(3,-24)$**	

$$f'(x) = (4)(x^2)(x-3)$$
$$f'(-1) = (+)\,(+)\,(-) \quad = \; -$$
$$f'(1) = (+)\,(+)\,(-) \quad = \; -$$
$$f'(4) = (+)\,(+)\,(+) \quad = \; +$$

Step 4: Skeleton graph for f; y-intercept

y-intercept = $f(0) = 3$, the constant term from the $f(x)$ rule.
(We already knew that $(0,3)$ was on the graph of f.)



Step 5: Find f'' and possible inflection numbers (PINs).

$$f'(x) = 4x^3 - 12x^2$$
$$f''(x) = 12x^2 - 24x$$
$$= 12x(x-2)$$

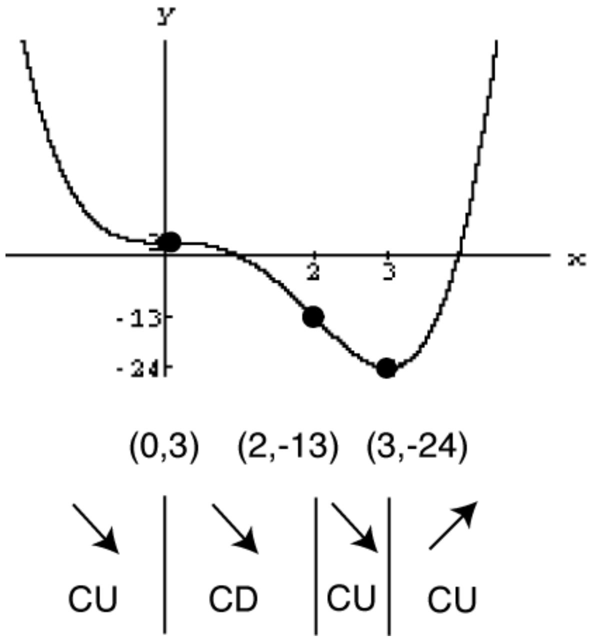
f'' is never DNE. It equals 0 at $x = 0$ and $x = 2$, which are in the domain of f.
The PINs are 0 and 2.

Step 6: Do a sign diagram for f'' and find inflection points (IPs).

	Test $x = -1$	**0**	Test $x = 1$	**2**	Test $x = 3$
f'' sign	+		–		+
f graph	CU ($\cup$)		CD ($\cap$)		CU ($\cup$)
Inflection Points (IPs)?		**Yes, IP**		**Yes, IP**	
Plug into $f(x)$ to get y		$(0, f(0))$ $\mathbf{(0,3)}$		$(2, f(2))$ $\mathbf{(2,-13)}$	

$$
\begin{aligned}
f''(x) &= (12)(x)(x-2) \\
f''(-1) &= (+)(-)(-) \quad = \quad + \\
f''(1) &= (+)(+)(-) \quad = \quad - \\
f''(3) &= (+)(+)(+) \quad = \quad +
\end{aligned}
$$

Step 7: Sketch the graph of f.



2) Find all critical numbers of $f(x) = \dfrac{1}{x^2+6x}$. Remember to clearly box in your answer(s). (6 points)

We must first find $f'(x)$.

Method 1: Quotient Rule

$$
\begin{aligned}
f'(x) &= \frac{\text{Lo} \cdot \text{D(Hi)} - \text{Hi} \cdot \text{D(Lo)}}{\text{the square of what's below}} \qquad \text{(Quotient Rule)} \\
&= \frac{\left(x^2+6x\right) \cdot D_x(1) - (1) \cdot D_x\left(x^2+6x\right)}{\left(x^2+6x\right)^2} \\
&= \frac{\overbrace{\left(x^2+6x\right) \cdot (0)}^{=0} - (1) \cdot (2x+6)}{\left(x^2+6x\right)^2} \\
&= \frac{-2x-6}{\left(x^2+6x\right)^2} \quad \text{or} \quad -\frac{2x+6}{\left(x^2+6x\right)^2}
\end{aligned}
$$

Method 2: Generalized Power Rule

$$f(x) = \left(x^2 + 6x\right)^{-1}$$
$$f'(x) = -\left(x^2 + 6x\right)^{-2} \cdot D_x\left(x^2 + 6x\right)$$
$$= -\left(x^2 + 6x\right)^{-2} \cdot (2x + 6)$$
$$= -\frac{2x + 6}{\left(x^2 + 6x\right)^2} \quad \text{or} \quad \frac{-2x - 6}{\left(x^2 + 6x\right)^2}$$

Notice: Both $f(x)$ and $f'(x)$ are rational, and they are both DNE (undefined) for those values of x that yield a zero denominator, and only those values. Any value of x that will make the denominator of $f'(x)$ equal to 0 will also make the denominator of $f(x)$ equal to 0. Therefore, the values of x that will make f' DNE (namely, 0 and -6) are <u>not</u> in the domain of f. They are <u>not</u> critical numbers.

Where is $f'(x) = 0$?

Set the numerator equal to 0 and solve for x.

$$-2x - 6 = 0$$
$$-2x = 6$$
$$x = -3$$

Note that -3 does not make the denominator of $f'(x)$ equal to 0.
-3 is in the domain of f, so -3 is a critical number.

$\mathbf{-3}$ is the only critical number of f.

3) True or False: If $f''(6) = 0$, then the graph of f must have an inflection point at $x = 6$. Circle one: (2 points)

True **False**

If the graph is not changing concavity (from concave up to concave down or vice-versa) around 6, then the point at 6 is not an inflection point. For example, $f(x) = (x - 6)^4$ has the property that $f''(6) = 0$, but $f''(x) = 12(x - 6)^2$ is positive both to the left and right of 6 on the number line, so the graph of f stays concave up to the left and to the right of 6.

4) True or False: The Second Derivative Test can be used to show that a point on the graph of f is neither a relative maximum point nor a relative minimum point. Circle one: (2 points)

True **False**

The Second Derivative Test can only be used to classify points as relative maximum points or relative minimum points.

5) Is the graph of $f(x)=\sqrt{x^3}$ concave up or concave down at $x=9$? Show work! (6 points)

$$\begin{aligned}
f(x) &= x^{3/2} \\
f'(x) &= \frac{3}{2}x^{1/2} \\
f''(x) &= \frac{3}{2}\cdot\frac{1}{2}x^{-1/2} \\
&= \frac{3}{4\sqrt{x}} \\
f''(9) &= \frac{3}{4\sqrt{9}} \quad \text{(You can see this is positive in value right now.)} \\
&= \frac{3}{4\cdot 3} \\
&= \frac{1}{4} \\
&> 0
\end{aligned}$$

f'' is positive in value at 9, so the graph is **concave up** at 9.

6) Find the absolute maximum and absolute minimum values of $f(x)=-3x^2+60x+7$ on the interval $[3,20]$ and label them "A.Max.Value" and "A.Min.Value." As you show your work, list all of the appropriate candidates for these values <u>and only those</u>, based on our discussion in class. (9 points)

The Extreme Value Theorem (EVT) applies, since f is continuous on the given closed interval.

Find any critical numbers:

$$\begin{aligned}
f(x) &= -3x^2+60x+7 \\
f'(x) &= -6x+60
\end{aligned}$$

f' is never DNE. Where is $f'(x)=0$?

$$\begin{aligned}
-6x+60 &= 0 \\
-6x &= -60 \\
x &= 10
\end{aligned}$$

10 is in $[3,20]$, so it is our sole critical number.

x	$f(x)$	Find highest, lowest values
3	160	
10	**307**	**A.Max.Value**
20	**7**	**A.Min.Value**

7) For the equation $xy^2 - 3x^5 + y^3 = 7$, use implicit differentiation to find $\frac{dy}{dx}$. (14 points)

Step 1: D_x each term.

$$D_x\left(xy^2\right) - D_x\left(3x^5\right) + D_x\left(y^3\right) = D_x(7)$$

We will need the Product Rule and the Chain Rule.

$$\begin{aligned}\left[D_x(x)\cdot y^2\right] + \left[x \cdot D_x\left(y^2\right)\right] - 15x^4 + 3y^2y' &= 0\\ \left[1\cdot y^2\right] + \left[x\cdot 2yy'\right] - 15x^4 + 3y^2y' &= 0\\ y^2 + 2xyy' - 15x^4 + 3y^2y' &= 0\end{aligned}$$

Step 2: Isolate the terms with y'.

$$2xyy' + 3y^2y' = 15x^4 - y^2$$

Step 3: Factor out y'.

$$y'\left(2xy + 3y^2\right) = 15x^4 - y^2$$

Step 4: Divide to isolate y', i.e., $\frac{dy}{dx}$.

$$y' \text{ or } \frac{dy}{dx} = \frac{\mathbf{15x^4 - y^2}}{\mathbf{2xy + 3y^2}}$$

8) Your company sells posters of Arnold Schwarzenegger. Fixed costs are \$300. Each poster costs \$4 to make. The price function is $p(x) = 346 - 3x$ in dollars, where x is the number of posters produced and sold. (20 points total)

a) Find the profit function, $P(x)$. Your answer must be in simplest form.

Find the revenue function, $R(x)$.

$$\begin{aligned}R(x) &= \begin{pmatrix}\text{Unit}\\ \text{price}\end{pmatrix}(\text{Quantity})\\ &= p(x)\cdot x\\ &= (346 - 3x)x\\ &= 346x - 3x^2\end{aligned}$$

Find the cost function, $C(x)$.

$$C(x) = 300 + 4x$$

Find the profit function, $P(x)$.

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= \left(346x - 3x^2\right) - (300 + 4x) \\ &= 346x - 3x^2 - 300 - 4x \\ &= \mathbf{-3x^2 + 342x - 300} \end{aligned}$$

b) How many posters should be produced? Verify that this gives us the absolute maximum profit, as we have done in class.

Find the critical numbers.

$$\begin{aligned} P(x) &= -3x^2 + 342x - 300 \\ P'(x) &= -6x + 342 \end{aligned}$$

P' is never DNE. Where is $P'(x) = 0$??

$$\begin{aligned} -6x + 342 &= 0 \\ -6x &= -342 \\ x &= 57 \end{aligned}$$

57 is in the domain of P, which is presumably $[0,\infty)$, so it is our sole critical number.

Verify we have an absolute maximum there.

Use the Second Derivative Test (remember, $P'(57) = 0$):

$$\begin{aligned} P'(x) &= -6x + 342 \\ P''(x) &= -6 \\ P''(57) &= -6 \end{aligned}$$

This is negative, so the graph of P is concave down $(\cap)$ at the corresponding point, which must be a <u>relative</u> maximum point.

It must also be an <u>absolute</u> maximum point, because P is continuous, and 57 is the only critical number.

57 posters should be produced.

c) What is your maximum profit?

$$\begin{aligned} P(x) &= -3x^2 + 342x - 300 \\ P(57) &= -3(57)^2 + 342(57) - 300 \\ &= \mathbf{\$9{,}447} \end{aligned}$$

9) A company's demand equation is $x = \sqrt{58 - p^2}$, where p is the price in dollars and x is the quantity consumers will demand at that price. Find $\frac{dp}{dx}$ when $p = 3$, and interpret your answer. (16 points)

For convenience, square both sides of the equation. Remember that x can't be negative.

$$x^2 = 58 - p^2$$

Since we want to find $\frac{dp}{dx}$, we will D_x each term.

$$D_x(x^2) = D_x(58) - D_x(p^2)$$

$$2x = 0 - 2p\underbrace{\frac{dp}{dx}}_{\text{tail}}$$

$$2x = -2p\frac{dp}{dx}$$

$$\frac{2x}{-2p} = \frac{dp}{dx}$$

$$-\frac{x}{p} = \frac{dp}{dx}$$

$$\frac{dp}{dx} = -\frac{x}{p}$$

(Below: If you didn't square.)

$$D_x(x) = D_x\left(\sqrt{58 - p^2}\right)$$

$$D_x(x) = D_x\left(58 - p^2\right)^{1/2}$$

$$1 = \frac{1}{2}\left(58 - p^2\right)^{-1/2} \cdot \underbrace{D_x\left(58 - p^2\right)}_{\text{tail}}$$

$$1 = \frac{1}{2}\left(58 - p^2\right)^{-1/2} \cdot \left(-2p\underbrace{\frac{dp}{dx}}_{\substack{2^{\text{nd}} \\ \text{tail}}}\right)$$

$$1 = -\frac{p\frac{dp}{dx}}{\sqrt{58 - p^2}}$$

$$-\sqrt{58 - p^2} = p\frac{dp}{dx}$$

$$-\frac{\sqrt{58 - p^2}}{p} = \frac{dp}{dx}$$

$$\frac{dp}{dx} = -\frac{\sqrt{58 - p^2}}{p} \quad \text{or} \quad -\frac{x}{p}$$

When $p = 3$, what is x?

$$x = \sqrt{58 - (3)^2}$$
$$= \sqrt{58 - 9}$$
$$= \sqrt{49}$$
$$= 7$$

Then,

$$\frac{dp}{dx} = -\frac{x}{p}$$
$$= -\frac{7}{3}$$
$$\approx \mathbf{-2.33} \left(\frac{\$}{\text{unit}}\right)$$

Interpretation:

When the unit price is \$3, the price must drop by about \$2.33 [per unit] if one more unit is to be demanded.