FINAL (SAMPLE SOLUTIONS)

USE A SCIENTIFIC CALCULATOR!

1) Approximate the area under the graph of $f(x) = \ln x$ from $a = 4$ to $b = 10$ by finding a Left Riemann Sum using 3 rectangles of the same width. Round off to four decimal places whenever you need to round off. (10 points)

Step 1: Find Δx, the width of each rectangle.

$$\Delta x = \frac{b-a}{n} = \frac{10-4}{3} = \frac{6}{3} = 2$$

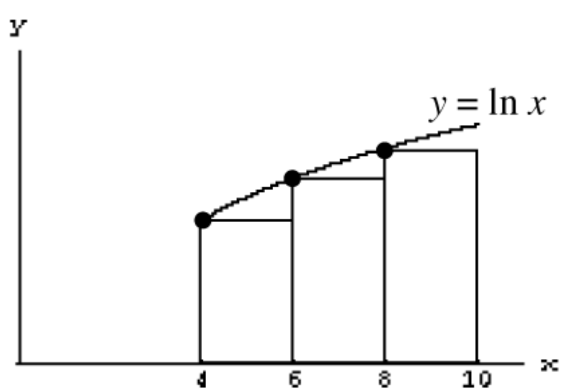
Step 2: Find the left breakpoints.

$$a = x_1 = 4 \xrightarrow{+2} x_2 = 6 \xrightarrow{+2} x_3 = 8$$

Step 3: Find the Left Riemann Sum.

$$\begin{aligned} & f(4)\cdot \Delta x + f(6)\cdot \Delta x + f(8)\cdot \Delta x \\ &= (\ln 4)(2) + (\ln 6)(2) + (\ln 8)(2) \\ &\approx 2.7726 + 3.5835 + 4.1589 \\ &= \mathbf{10.5150 \ \ square\ units} \end{aligned}$$

Note: The exact value is closer to 11.4807.



2) Find the integrals. Simplify wherever possible. (32 points total)

a) $\int_1^5 \underbrace{\left(3x^{-1} - 3x^2\right)}_{\text{cont. on } [1,5]} dx$ (6 points)

$$= \left[3\ln x - x^3\right]_1^5 \qquad (\ln|x| = \ln x \text{ on } [1,5].)$$
$$= \left[3\ln 5 - (5)^3\right] - \left[3\ln 1 - (1)^3\right]$$
$$= [3\ln 5 - 125] - [0 - 1]$$
$$= 3\ln 5 - 125 + 1$$
$$= \mathbf{3\ln 5 - 124}$$
$$\textbf{or } \mathbf{\ln 5^3 - 124} \textbf{ or } \mathbf{\ln 125 - 124}$$

b) $\int x\left(x^2 - 4\right)^6 dx$ (6 points)

$$u = x^2 - 4$$
$$du = 2x\, dx$$

$$= \frac{1}{2}\int 2x\left(x^2 - 4\right)^6 dx$$
$$= \frac{1}{2}\int u^6\, du$$
$$= \frac{1}{2}\left(\frac{u^7}{7}\right) + C$$
$$= \frac{1}{14}u^7 + C$$
$$= \boldsymbol{\frac{1}{14}\left(x^2 - 4\right)^7 + C}$$

c) $\int e^{x^3+6x-1}\left(x^2 + 2\right) dx$ (6 points)

$$u = x^3 + 6x - 1$$
$$du = 3x^2 + 6$$
$$= 3\left(x^2 + 2\right) dx$$

$$= \frac{1}{3}\int e^{x^3+6x-1} \cdot 3\left(x^2 + 2\right) dx$$
$$= \frac{1}{3}\int e^u\, du$$
$$= \frac{1}{3}e^u + C$$
$$= \boldsymbol{\frac{1}{3}e^{x^3+6x-1} + C}$$

d) $\int \frac{\ln x}{x}\,dx$ (6 points)

$$u = \ln x$$
$$du = \frac{1}{x}\,dx$$

$$= \int (\ln x)\cdot\frac{1}{x}\,dx$$
$$= \int u\,du$$
$$= \frac{u^2}{2} + C$$
$$= \boldsymbol{\frac{(\ln x)^2}{2} + C}$$

e) $\int_1^2 \underbrace{\frac{x^2}{x^3+4}}_{\text{cont. on } [1,2]}\,dx$ (8 points)

$$u = x^3 + 4$$
$$du = 3x^2\,dx$$

New limits:

$$x = 1 \Rightarrow u = (1)^3 + 4 = 5$$
$$x = 2 \Rightarrow u = (2)^3 + 4 = 12$$

$$= \frac{1}{3}\int_{x=1}^{x=2} \frac{3x^2}{x^3+4}\,dx$$
$$= \frac{1}{3}\int_{u=5}^{u=12} \frac{du}{u}$$
$$= \frac{1}{3}[\ln u]_5^{12} \qquad (\ln|u| = \ln u \text{ on } [5,12].)$$
$$= \boldsymbol{\frac{1}{3}(\ln 12 - \ln 5)}$$

or $\boldsymbol{\ln \sqrt[3]{\frac{12}{5}}}$ (you could also rationalize the denominator)

3) The weight of a blob increases at the rate of $0.3e^{0.2t}$ pounds per day, where t is measured in days. Find the total increase in the blob's weight from $t = 3$ to $t = 6$. (6 points)

$$\int_3^6 0.3e^{0.2t}\,dt = \left[\frac{0.3e^{0.2t}}{0.2}\right]_3^6$$
$$= \left[\frac{3}{2}e^{0.2t}\right]_3^6$$
$$= \frac{3}{2}e^{0.2(6)} - \frac{3}{2}e^{0.2(3)}$$
$$= \boldsymbol{\frac{3}{2}\left(e^{1.2} - e^{0.6}\right)} \textbf{ pounds}$$

4) Find the average value of $f(x)=x^3$ on the interval $[0,3]$. (6 points)

$$f_{av}=\frac{\int_a^b f(x)\,dx}{b-a}$$

$$=\frac{\int_0^3 x^3}{3-0}$$

$$=\frac{\left[\frac{x^4}{4}\right]_0^3}{3}$$

$$=\frac{\frac{(3)^4}{4}-0}{3} \quad \left(=\frac{(3)^4}{4\cdot 3}=\frac{(3)^3}{4}\right)$$

$$=\mathbf{\frac{27}{4}}$$

5) Find the area bounded by the graphs of $y=5x^2+x-11$ and $y=3x^2-3x+5$. (16 points)

We need to set up: $\int_a^b [(\text{top})-(\text{bottom})]\,dx$.

<u>Step 1</u>: Where are the intersection points?

Solve $\begin{cases} y=5x^2+x-11 \\ y=3x^2-3x+5 \end{cases}$ for x, at least.

$$5x^2+x-11=3x^2-3x+5$$
$$2x^2+4x-16=0$$
$$2(x^2+2x-8)=0$$
$$2(x+4)(x-2)=0$$

The intersection points are at $x=-4$ and $x=2$.

<u>Step 2</u>: Who's on top?

Test $x=0$, since 0 is between -4 and 2.

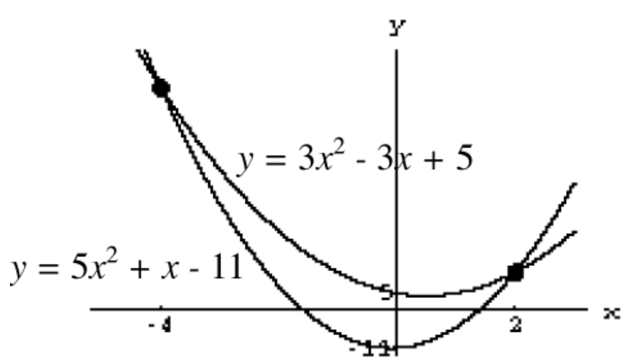
$$y=5x^2+x-11 \xrightarrow{x=0} y=-11$$
$$y=3x^2-3x+5 \xrightarrow{x=0} y=5$$

The graph of the second equation is on top.

<u>Step 3</u>: Set up the definite integral.

$$\int_{-4}^{2}\left[\left(3x^2-3x+5\right)-\left(5x^2+x-11\right)\right]dx$$
$$=\int_{-4}^{2}\left[3x^2-3x+5-5x^2-x+11\right]dx$$
$$=\int_{-4}^{2}\left[-2x^2-4x+16\right]dx$$
$$=\left[-2\left(\frac{x^3}{3}\right)-4\left(\frac{x^2}{2}\right)+16x\right]_{-4}^{2}$$
$$=\left[-\frac{2}{3}x^3-2x^2+16x\right]_{-4}^{2}$$
$$=\left[-\frac{2}{3}(2)^3-2(2)^2+16(2)\right]-\left[-\frac{2}{3}(-4)^3-2(-4)^2+16(-4)\right]$$
$$=\left[-\frac{2}{3}(8)-2(4)+32\right]-\left[-\frac{2}{3}(-64)-2(16)-64\right]$$
$$=\left[-\frac{16}{3}-8+32\right]-\left[\frac{128}{3}-32-64\right]$$
$$=\left[-\frac{16}{3}+24\right]-\left[\frac{128}{3}-96\right]$$
$$=-\frac{16}{3}+24-\frac{128}{3}+96$$
$$=-\frac{144}{3}+120$$
$$=-48+120$$
= **72 square units**

<u>Note</u>: Here is a graph of the region of interest:



6) Find the domain of $f(x,y)=\frac{\ln x}{y}$. (2 points)

$$\left\{(\boldsymbol{x},\boldsymbol{y})\,\middle|\,\boldsymbol{x}>\mathbf{0},\ \boldsymbol{y}\neq\mathbf{0}\right\}$$

7) Let $f(x,y)=x^2y^3+e^{xy}$. (6 points total)

a) Find $f_x(x,y)$.

$$f(x,y)=x^2\overbrace{y^3}^{\text{"\#"}}+e^{x\overbrace{y}^{\text{"\#"}}}$$

$$f_x(x,y)=2x\cdot y^3+e^{xy}\cdot\underbrace{D_x(xy)}_{=y}$$

$$=\mathbf{2xy^3+ye^{xy}}$$

b) Find $f_x(3,1)$.

$$f_x(3,1)=2(3)(1)^3+(1)e^{(3)(1)}$$
$$=\mathbf{6+e^3}$$

8) Let $f(x,y,z)=x\ln\left(2x^3+4y\right)+z^2$. Find $f_y(x,y,z)$. (5 points)

$$f(x,y,z)=\underbrace{x}_{\text{"\#"}}\ln\left(\underbrace{2x^3}_{\text{"\#"}}+4y\right)+\underbrace{z^2}_{\text{"\#"}}$$

$$f_y(x,y,z)=x\cdot\frac{1}{2x^3+4y}\cdot D_y\left(\underbrace{2x^3}_{\text{"\#"}}+4y\right)+0$$

$$=x\cdot\frac{1}{2x^3+4y}\cdot(4)$$

$$=\frac{\mathbf{4x}}{\mathbf{2x^3+4y}}\quad\textbf{or}\quad\frac{\mathbf{2x}}{\mathbf{x^3+2y}}$$

9) Let $f(x,y)=2x^2-6xy-14x+3y^2+18y+7$. Find any critical points, and classify each critical point (as corresponding to a Relative Maximum Point, a Relative Minimum Point, or Neither). Find any relative extreme values. (17 points total) Note: We may or may not cover Section 7.3 in our class.

Step 1: Find any critical points.

$$f_x(x,y)=\ \ 4x-6y-14\overset{\text{Set}}{=}0$$
$$f_y(x,y)=-6x+6y+18\overset{\text{Set}}{=}0$$

Solve the system $\begin{cases} 4x-6y-14=0 \\ -6x+6y+18=0 \end{cases}$

$$\begin{cases} 4x-6y=14 \\ -6x+6y=-18 \end{cases}$$

$$\begin{aligned} -2x &= -4 \\ x &= 2 \end{aligned}$$

Plug $x=2$ into a previous equation with *x* and *y*.
For example,

$$\begin{aligned} 4x-6y &= 14 \\ 4(2)-6y &= 14 \\ 8-6y &= 14 \\ -6y &= 6 \\ y &= -1 \end{aligned}$$

The only critical point is $(2,-1)$.
(This point is in the domain of *f*.)

<u>Step 2</u>: Find *D*.

Remember,

$$f_x(x,y) = 4x-6y-14$$
$$f_y(x,y) = -6x+6y+18$$

$$\begin{aligned} A &= f_{xx}(x,y) = 4 \\ B &= f_{xy}(x,y) = -6 \\ C &= f_{yy}(x,y) = 6 \\ D &= AC-B^2 \\ &= (4)(6)-(-6)^2 \\ &= 24-36 \\ &= -12 \end{aligned}$$

<u>Step 3</u>: Classify the critical point.

$D=-12<0$ at all points (x,y), including the critical point.
Since $D<0$ there, the corresponding point is **Neither** (i.e., it is a "saddle point.")

<u>Step 4</u>: Find any relative extreme values.

There aren't any relative extreme values.