QUIZ #3 (SAMPLE SOLUTIONS)

USE A SCIENTIFIC CALCULATOR!

1) For parts a) and b), sketch the graph of f. You must:

- Find and label all critical points and inflection points (if any).
- Classify all critical points as relative maximum points, relative minimum points, or neither.
- Find the y-intercept.
- Have your graph correctly show where f is increasing / decreasing, and where f is concave up / concave down.
- Show all steps, as we have done in class.

(44 points total)

a) $f(x) = 2x^3 + 9x^2 - 24x + 20$. (26 points)

Step 1: Domain = **R**.

Step 2: Find f' and critical numbers (CNs).

$$\begin{aligned} f(x) &= 2x^3 + 9x^2 - 24x + 20 \\ f'(x) &= 6x^2 + 18x - 24 \\ &= 6(x^2 + 3x - 4) \\ &= 6(x+4)(x-1) \end{aligned}$$

This is never DNE, but it equals 0 at $x = -4$ and $x = 1$, which are in the domain of f.

The CNs are -4 and 1.

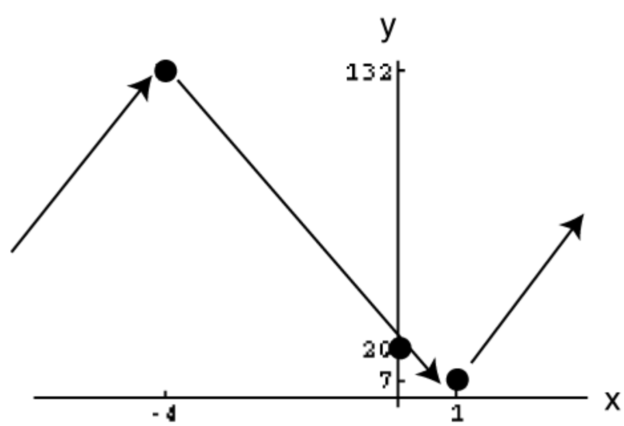
Step 3: Do a sign diagram for f' and classify critical points (CPs).

	Test $x = -5$	**−4**	Test $x = 0$	**1**	Test $x = 2$
f' sign	+		−		+
f	↗		↘		↗
Classify CPs		**R.Max. Pt.**		**R.Min. Pt.**	
Plug into $f(x)$ to get y		$(-4, f(-4))$ $\mathbf{(-4,132)}$		$(1, f(1))$ $\mathbf{(1,7)}$	

$$\begin{aligned} f'(x) &= (6)(x+4)(x-1) \\ f'(-5) &= (+)(-)(-) \quad = \quad + \\ f'(0) &= (+)(+)(-) \quad = \quad - \\ f'(2) &= (+)(+)(+) \quad = \quad + \end{aligned}$$

Step 4: Skeleton graph for f; y-intercept

y-intercept = $f(0) = 20$, the constant term from the $f(x)$ rule.



Step 5: Find f'' and possible inflection numbers (PINs).

$$\begin{aligned} f'(x) &= 6x^2 + 18x - 24 \\ f''(x) &= 12x + 18 \\ &= 6(2x+3) \end{aligned}$$

Note:

$$\begin{aligned} 2x + 3 &= 0 \\ 2x &= -3 \\ x &= -\frac{3}{2} \text{ or } -1.5 \end{aligned}$$

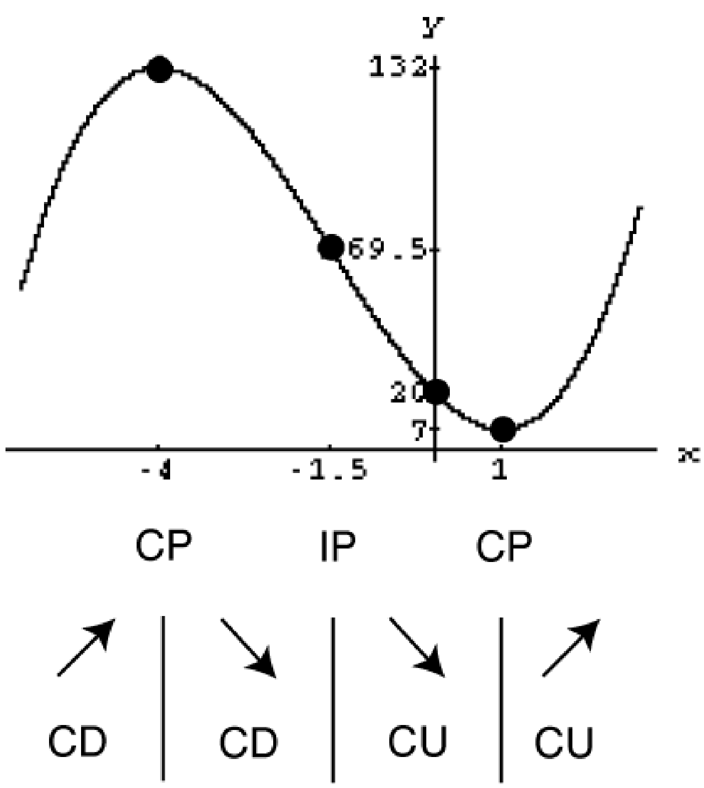
f'' is never DNE, but it equals 0 at $x = -1.5$, which is in the domain of f. The sole PIN is -1.5.

Step 6: Do a sign diagram for f'' and find inflection points.

	Test $x = -2$	$\mathbf{-1.5}$	Test $x = 0$
f'' sign	$-$		$+$
f graph	CD $(\cap)$		CU $(\cup)$
Inflection Points (IPs)?		**Yes, IP**	
Plug into $f(x)$ to get y		$(-1.5, f(-1.5))$ $(\mathbf{-1.5, 69.5})$	

$$f''(x) = (6)(2x+3)$$
$$f''(-2) = (+)(-) \quad = \quad -$$
$$f''(0) = (+)(+) \quad = \quad +$$

Step 7: Sketch the graph of f.



b) $f(x)=\sqrt[5]{x^4}$. (18 points)

Step 1: Domain = **R**.

(Odd roots don't cause the trouble that even roots do.)

Step 2: Find f' and critical numbers (CNs).

$$f(x)=x^{4/5}$$
$$f'(x)=\frac{4}{5}x^{-1/5}$$
$$=\frac{4}{5x^{1/5}}$$
$$=\frac{4}{5\left(\sqrt[5]{x}\right)}$$

This is never 0, but it is DNE at $x=0$, which is in the domain of f.
The sole CN is 0.

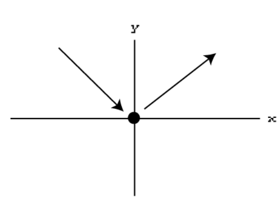
Step 3: Do a sign diagram for f' and classify critical points (CPs).

	Test $x=-1$	**0**	Test $x=1$
f' sign	$-$		$+$
f	↘		↗
Classify CPs		**R.Min. Pt.**	
Plug into $f(x)$ to get y		$(0,f(0))$ **(0,0)**	

$$f'(x)=\frac{4}{5\left(\sqrt[5]{x}\right)}$$
$$f'(-1)=\frac{4}{5\left(\sqrt[5]{-1}\right)} \quad = \quad -$$
$$f'(1)=\frac{4}{5\left(\sqrt[5]{1}\right)} \quad = \quad +$$

Step 4: Skeleton graph for f; y-intercept

y-intercept = $f(0)=0$



Step 5: Find f'' and possible inflection numbers (PINs).

$$f'(x) = \frac{4}{5}x^{-1/5}$$

$$f''(x) = \left(\frac{4}{5}\right)\left(-\frac{1}{5}\right)x^{-6/5}$$

$$= -\frac{4}{25}x^{-6/5}$$

$$= -\frac{4}{25x^{6/5}}$$

$$= -\frac{4}{25\left(\sqrt[5]{x}\right)^6}$$

This is never 0, but it is DNE at $x = 0$, which is in the domain of f.
The sole PIN is 0.

Step 6: Do a sign diagram for f'' and find inflection points.

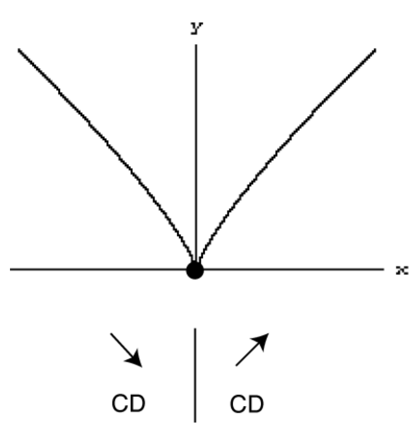
	Test $x = -1$	**0**	Test $x = 1$
f'' sign	–		–
f graph	CD $(\cap)$		CD $(\cap)$
Inflection Points (IPs)?		**No**	

$$f''(x) = -\frac{4}{25\left(\sqrt[5]{x}\right)^6}$$

$$f''(-1) = -\frac{4}{25\underbrace{\left(\sqrt[5]{-1}\right)^6}_{=1}} = -$$

$$f''(1) = -\frac{4}{25\left(\sqrt[5]{1}\right)^6} = -$$

Step 7: Sketch the graph of f.



2) Find the absolute maximum and absolute minimum values of $f(x)=2x^2-12x+5$ on the interval $[1,4]$. (8 points)

The Extreme Value Theorem (EVT) applies, since f is continuous on the given closed interval.

Find any critical numbers:

$$f(x)=2x^2-12x=5$$
$$f'(x)=4x-12$$

Note:

$$4x-12=0$$
$$4x=12$$
$$x=3$$

f' is never DNE, but it equals 0 at $x=3$, which is in $[1,4]$.
The sole critical number is 3.

x	$f(x)$	Find highest, lowest values
1	**−5**	**Absolute maximum value**
3	**−13**	**Absolute minimum value**
4	−11	

3) Your company sells TVs. Fixed costs are \$2000. Each TV costs \$100 to make. The price function is $p(x)=1100-20x$ in dollars, where x is the number of TVs produced and sold. (20 points total)

a) Find the profit function, $P(x)$.

Find the revenue function, $R(x)$.

$$R(x)=\begin{pmatrix}\text{Unit}\\ \text{price}\end{pmatrix}(\text{Quantity})$$
$$=p(x)\cdot x$$
$$=(1100-20x)x$$
$$=1100x-20x^2$$

Find the cost function, $C(x)$.

$$C(x)=2000+100x$$

Find the profit function, $P(x)$.

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= \left(1100x - 20x^2\right) - (2000 + 100x) \\ &= 1100x - 20x^2 - 2000 - 100x \\ &= \mathbf{-20x^2 + 1000x - 2000} \end{aligned}$$

b) How many TVs should be produced? Verify that this gives us the absolute maximum profit.

Find the critical numbers.

$$\begin{aligned} P(x) &= -20x^2 + 1000x - 2000 \\ P'(x) &= -40x + 1000 \end{aligned}$$

Note:

$$\begin{aligned} -40x + 1000 &= 0 \\ -40x &= -1000 \\ x &= 25 \end{aligned}$$

P' is never DNE, but it equals 0 at $x = 25$, which is in the domain of P (which is presumably $[0,\infty)$.)

The sole critical number is 25.

Verify we have an absolute maximum there.

Use the Second Derivative Test (remember, $P'(25) = 0$):

$$\begin{aligned} P'(x) &= -40x + 1000 \\ P''(x) &= -40 \\ P''(25) &= -40 \end{aligned}$$

This is negative, so the graph of P is concave down $(\cap)$ at the critical point, which must be a relative maximum point.

It must also be an absolute maximum point, because P is continuous, and 25 is the only critical number.

25 TVs should be produced.

c) What is your maximum profit?

$$\begin{aligned} P(x) &= -20x^2 + 1000x - 2000 \\ P(25) &= -20(25)^2 + 1000(25) - 2000 \\ &= \mathbf{\$10{,}500} \end{aligned}$$

4) For the equation $x^3y + y^2 = 4$, use implicit differentiation to find $\frac{dy}{dx}$. (10 points)

Step 1: D_x each term.

$$D_x(x^3y) + D_x(y^2) = D_x(4)$$

We will need the Product Rule and the Chain Rule.

$$\left[D_x(x^3)\cdot y\right] + \left[x^3 \cdot D_x(y)\right] + 2yy' = 0$$
$$3x^2y + x^3y' + 2yy' = 0$$

Step 2: Isolate the terms with y'.

$$x^3y' + 2yy' = -3x^2y$$

Step 3: Factor out y'.

$$y'(x^3 + 2y) = -3x^2y$$

Step 4: Divide to isolate y'.

$$y' = -\frac{\mathbf{3x^2y}}{\mathbf{x^3 + 2y}}$$

5) Darrell's company sells car alarms. Its revenue (R) is given by $R = 4x^2 + 300x$, where x is the population of Darrell's hometown. If the population of Darrell's hometown grows at the rate of 20 people per week, find how rapidly revenue is growing when the population of the town is 500. (12 points)

Define variables, and organize info:

Let $t =$ time in weeks.
Given: $\frac{dx}{dt} = 20$.
Find: $\frac{dR}{dt}$ when $x = 500$.

We will D_t each term.

$$R = 4x^2 + 300x$$
$$D_t(R) = D_t(4x^2) + D_t(300x)$$
$$\frac{dR}{dt} = 8x\frac{dx}{dt} + 300\frac{dx}{dt}$$

Plug in: $\frac{dx}{dt} = 20$ and $x = 500$.

$$\begin{aligned}\frac{dR}{dt} &= 8x\frac{dx}{dt} + 300\frac{dx}{dt} \\ &= 8(500)(20) + 300(20) \\ &= \mathbf{86{,}000\ \frac{\$}{week}}\end{aligned}$$

Interpretation (optional):

The revenue is growing at the rate of $86,000 per week when the population is 500.

6) If 3 is a critical number of f, $f'(3) = 0$, and $f''(3)$ is positive, then what kind of a point does f have at $x = 3$? Circle one: (3 points)

Relative maximum point **Relative minimum point** Neither

Use the Second Derivative Test. If f'' is positive at a critical point, then the graph of f is concave up $(\cup)$ there. The critical point is a relative minimum point.

7) True or False: If 7 is a critical number of f, $f'(7) = 0$, and $f''(7) = 0$, then f must have neither a relative maximum point, nor a relative minimum point at $x = 7$. Circle one: (3 points)

True **False**

The Second Derivative Test is useless (i.e., it gives no information) when f'' is 0.