Math 141 Name: ________________________

FINAL – SAMPLE CH.10 PROBLEMS

MATH 141 – FALL 2017 – KUNIYUKI

1) An ellipse has equation $4x^2 + y^2 + 24x - 4y + 24 = 0$ in the usual *xy*-plane. (26 points total)

   a) Find the standard form of the equation of this ellipse. (11 points)

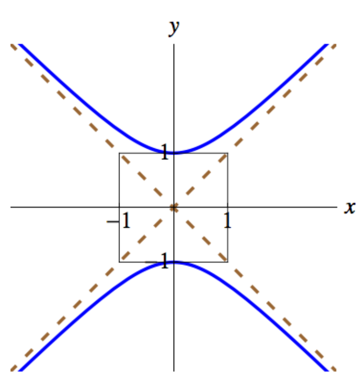
   b) The center of this ellipse is at what point? (2 points)

   c) The vertices of this ellipse are at what points? (4 points)

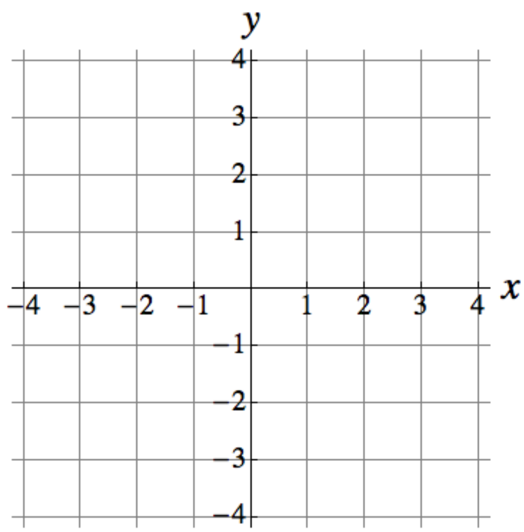
   d) The foci of this ellipse are at what points? (6 points)

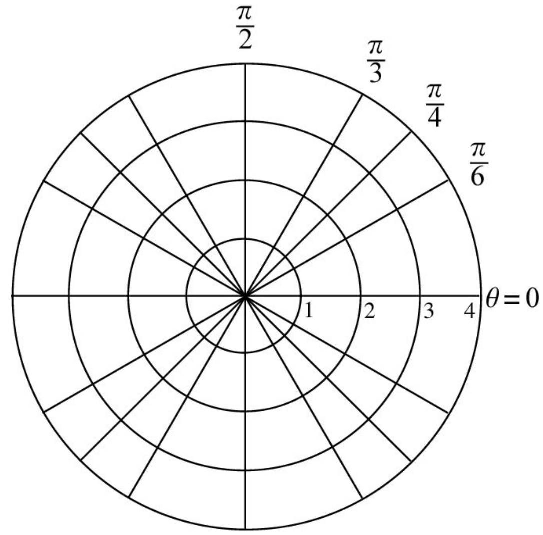
   e) What is the eccentricity of this ellipse? (3 points)

2) The graph below is the graph of (box in one:) $x^2 - y^2 = 1$ $y^2 - x^2 = 1$ (3 points)



3) Sketch the graph of the polar equation $r = 2\cos(\theta)$, where r and θ are polar coordinates. You may use either the Cartesian or polar graph paper below; box in the one you use. Use arrows to indicate orientation. (10 points)





SOLUTIONS

1) An ellipse has equation $4x^2 + y^2 + 24x - 4y + 24 = 0$ in the usual *xy*-plane. (26 points total)

a) Find the standard form of the equation of this ellipse. (11 points)

$$4x^2 + y^2 + 24x - 4y + 24 = 0$$

$$\left(4x^2 + 24x\right) + \left(y^2 - 4y\right) = -24 \qquad \text{(Group terms.)}$$

$$4\left(x^2 + 6x\right) + \left(y^2 - 4y\right) = -24 \qquad \text{(Factor out leading coefficients.)}$$

$$4\left(x^2 + 6x + 9\right) + \left(y^2 - 4y + 4\right) = -24 + 4(9) + (4) \qquad \text{(CTS and balance.)}$$

$$4(x+3)^2 + (y-2)^2 = 16 \qquad \text{(Factor PSTs.)}$$

$$\frac{(x+3)^2}{\frac{16}{4}} + \frac{(y-2)^2}{16} = 1$$

(Divide both sides by 16; then, 1 is on the right side.)

$$\boxed{\frac{(x+3)^2}{4} + \frac{(y-2)^2}{16} = 1}$$

b) The center of this ellipse is at what point? (2 points)

The center is at: $\boxed{(-3, 2)}$. (Think: What makes the left side equal to 0?)

c) The vertices of this ellipse are at what points? (4 points)

- $a^2 = 16$, the larger denominator, so $a = 4$. This is the distance between the center and each vertex.
- The larger denominator is part of the term with *y*, so the ellipse is "*y*-long" (vertical).
- Therefore, the vertices lie 4 units above and below the center.

The vertices are at: $\boxed{(-3, 6) \text{ and } (-3, -2)}$.

d) The foci of this ellipse are at what points? (6 points)

- $a^2 = 16$, and $b^2 = 4$, which is the smaller denominator.
- Find *c*.

$$c^2 = a^2 - b^2 = 16 - 4 = 12 \Rightarrow$$

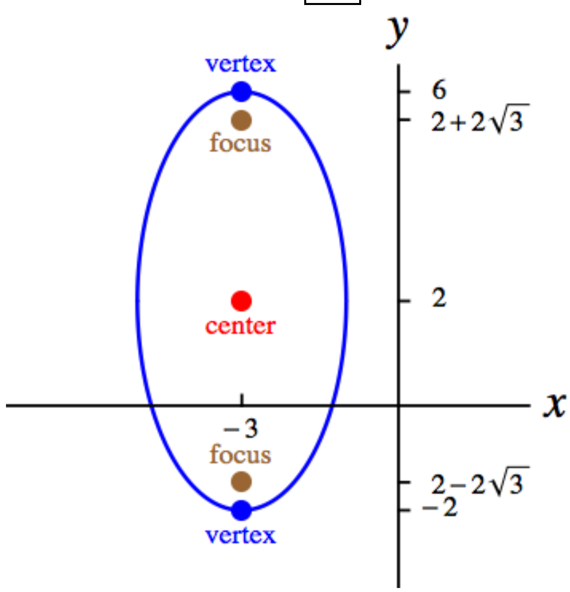
$$c = \sqrt{12} \text{ (Take the positive root.)} = \sqrt{4 \cdot 3} = 2\sqrt{3}$$

- *c* is the distance between the center and each focus.
- Because the ellipse is "*y*-long" (vertical), the foci lie *c* units above and below the center.

The foci are at: $\boxed{\left(-3, 2 + 2\sqrt{3}\right) \text{ and } \left(-3, 2 - 2\sqrt{3}\right)}$. $2 + 2\sqrt{3} \approx 5.46$; $2 - 2\sqrt{3} \approx -1.46$

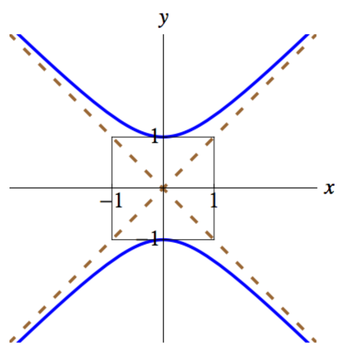
e) What is the eccentricity of this ellipse? (3 points)

$$e = \frac{c}{a} = \frac{2\sqrt{3}}{4} = \boxed{\frac{\sqrt{3}}{2}} \; (\approx 0.866)$$



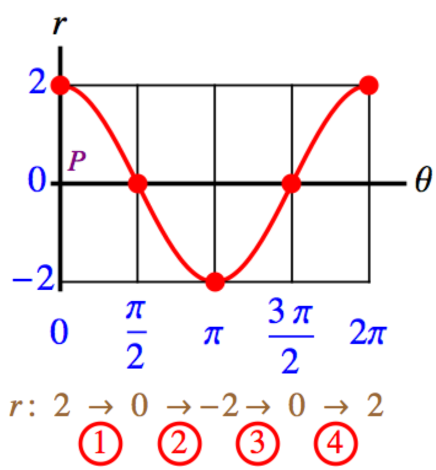
2) The graph below is the graph of (box in one:) $x^2 - y^2 = 1$ $\boxed{y^2 - x^2 = 1}$

(3 points)



3) Sketch the graph of the polar equation $r = 2\cos(\theta)$, where r and θ are polar coordinates. You may use either the Cartesian or polar graph paper below; box in the one you use. Use arrows to indicate orientation. (10 points)

First, graph r against θ as Cartesian coordinates. Graph one cycle of $r = 2\cos(\theta)$.



Now, graph r and θ as polar coordinates. Draw in Quadrant …

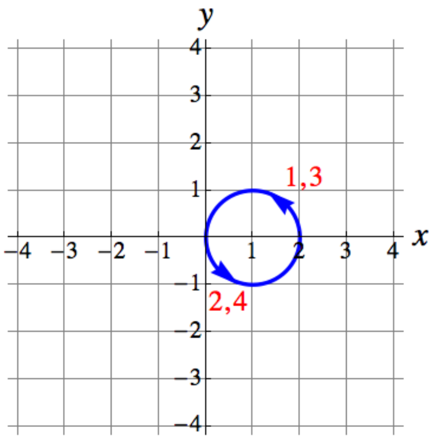
As $\theta:\ 0 \to \frac{\pi}{2},\ \ r:\ 2 \to 0.$ (Phase 1) I

As $\theta:\ \frac{\pi}{2} \to \pi,\ \ r:\ 0 \to -2.$ (Phase 2) IV (not II, because $r \le 0$)

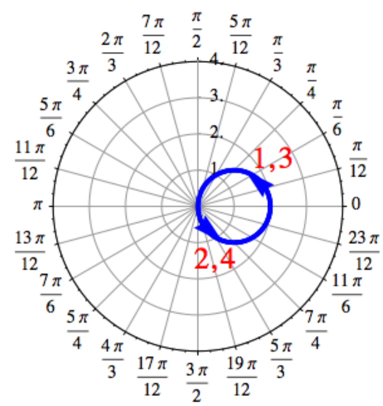
As $\theta:\ \pi \to \frac{3\pi}{2},\ \ r:\ -2 \to 0.$ (Phase 3) I (not III, because $r \le 0$)

As $\theta:\ \frac{3\pi}{2} \to 2\pi,\ \ r:\ 0 \to 2.$ (Phase 4) IV

Using Cartesian graph paper:



Using polar graph paper:



Obtaining the equation of the circle in Cartesian coordinates:

$$r = 2\cos(\theta) \quad \Rightarrow \quad (\text{Multiply both sides by } r.)$$
$$r^2 = 2r\cos(\theta)$$

We need not exclude the case $r = 0$, since 0 is in the range of the $2\cos(\theta)$ function. The pole (origin), which corresponds to $r = 0$, lies on the graph.

$$x^2 + y^2 = 2x$$
$$x^2 - 2x + y^2 = 0$$
$$\left(x^2 - 2x + 1\right) + y^2 = 0 + 1 \quad (\text{CTS and balance.})$$
$$(x-1)^2 + y^2 = 1$$

We have a circle of radius 1 centered at $(1, 0)$.