CHAPTER 5:

Analytic Trigonometry

(A) means "refer to Part A," (B) means "refer to Part B," etc.
(Calculator) means "use a calculator." Otherwise, do not use a calculator.

Write units in your final answers where appropriate. Try to avoid rounding intermediate results; if you do round off, do it to at least five significant digits.

SECTION 5.1: FUNDAMENTAL TRIGONOMETRIC IDENTITIES

Ignore domain issues in these problems.

1) Complete the Identities. Fill out the table below so that, for each row, the left side is equivalent to the right side, based on the type of identity given in the last column. (A)

Left Side	Right Side	Type of Identity (ID)
$\csc(x)$		Reciprocal ID
$\tan(x)$		Reciprocal ID
$\tan(x)$		Quotient ID
$\tan\left(\frac{\pi}{2}-x\right)$		Cofunction ID
$\cos(x)$		Cofunction ID
$\sin(-x)$		Even / Odd (Negative-Angle) ID
$\cos(-x)$		Even / Odd (Negative-Angle) ID
$\tan(-x)$		Even / Odd (Negative-Angle) ID
$\sin^2(x)+\cos^2(x)$		Pythagorean ID
$\tan^2(x)+1$		Pythagorean ID
$1+\cot^2(x)$		Pythagorean ID

2) Simplify the following. Find the most "compact" equivalent expression. (A-F)

a) $\dfrac{1-\sec(-x)}{1-\cos(-x)}$; b) $\dfrac{\tan(\theta)+\cot(\theta)}{\cot(\theta)}$; c) $\sec^2(x)-\sec^2(x)\sin^2(x)$;

d) $\cot^4(x)+2\cot^2(x)+1$; e) $\dfrac{\sin\left(\frac{\pi}{2}-t\right)}{\cot(t)}$; f) $\left[\csc(\alpha)+\cot(-\alpha)\right]\left[1+\cos(-\alpha)\right]$

3) Use the given trigonometric substitution to rewrite the given algebraic expression as a trigonometric expression in θ, where θ is acute. Simplify. (These types of substitutions are used in an advanced integration technique in calculus.) (G)

a) Substitute $x=4\sin(\theta)$ in the expression $\sqrt{16-x^2}$.

b) Substitute $x=6\tan(\theta)$ in the expression $\sqrt{x^2+36}$.

c) Substitute $x=3\sec(\theta)$ in the expression $\sqrt{x^2-9}$.

SECTION 5.2: VERIFYING TRIGONOMETRIC IDENTITIES

Ignore domain issues in these problems.

1) Verifying the following identities. (A-C)

a) $\dfrac{1-\sin(-x)}{\cos(-x)}=\sec(x)+\tan(x)$

b) $\dfrac{\sin(u)\cos(u)-\cos(u)}{\sin(u)-\sin^2(u)}=-\cot(u)$

c) $\dfrac{1}{\tan(\alpha)}-\tan(-\alpha)=\csc(\alpha)\sec(\alpha)$

d) $\dfrac{1}{\csc(x)+1}+\dfrac{1}{\csc(x)-1}=2\sec(x)\tan(x)$

e) $\sqrt{\dfrac{1-\cos(\theta)}{1+\cos(\theta)}}=\dfrac{1-\cos(\theta)}{\left|\sin(\theta)\right|}$

f) $\dfrac{1}{1-\cos(\beta)}=\csc^2(\beta)+\csc(\beta)\cot(\beta)$

g) $\tan^{5/2}(x)+\tan^{1/2}(x)=\left[\sec^2(x)\right]\sqrt{\tan(x)}$

SECTION 5.3: SOLVING TRIGONOMETRIC EQUATIONS

Use radian measure for angles in the following problems. Give exact solutions.

1) Find all real solutions, and find the particular solutions in the interval $[0, 2\pi)$.

a) $2\sin(x)-\sqrt{3}=0$. (A)

b) $6\cos(\theta)+3\sqrt{2}=0$. (A)

c) $\cos(\alpha)=-\pi$. (A)

d) $\sin(u)=-1$. (A)

e) $\cos(u)=0$. (A)

f) $2+\csc(u)=0$. (A)

g) $\sec(x)=2$. (A)

h) $\csc(x)=\frac{1}{2}$. (A)

i) $3\tan(x)=\sqrt{3}$. (A, B)

j) $\cot(\theta)=0$. (A, B)

k) $\cot^2(\theta)=3$. (A-C)

l) $\tan^2(\theta)+\tan(\theta)=0$. (A, B, D)

m) $\csc^4(x)-3\csc^3(x)+2\csc^2(x)=0$. (A, D)

n) $2\cos^3(x)+\sin^2(x)=1+\cos(-x)$. (A, D, E)

o) $2\cos(4x)-1=0$. (A, F)

p) $\csc(3x)=1$. (A, F)

q) $\tan^2(3x)=3$. (B, C, F)

2) Consider the equation $\tan(x)=2$. (A, B, G)

a) Find the solutions of the equation in the interval $[0, 2\pi)$.

b) Approximate the solutions you found in a) to four significant digits. (Calculator)

c) Find all real solutions of the equation.

3) Consider the equation $5\cos^2(x)-14\cos(x)-3=0$. (A, D, G)

a) Find the solutions of the equation in the interval $[0, 2\pi)$.

b) Approximate the solutions you found in a) to four significant digits. (Calculator)

c) Find all real solutions of the equation.

SECTIONS 5.4 and 5.5: MORE TRIGONOMETRIC IDENTITIES

1) Complete the Identities. Fill out the table below so that, for each row, the left side is equivalent to the right side, based on the type of identity given in the last column. (A, Handout)

Left Side	Right Side	Type of Identity (ID)
$\sin(u+v)$		Sum ID
$\cos(u+v)$		Sum ID
$\tan(u+v)$		Sum ID
$\sin(u-v)$		Difference ID
$\cos(u-v)$		Difference ID
$\tan(u-v)$		Difference ID
$\sin(2u)$		Double-Angle ID
$\cos(2u)$		Double-Angle ID (write <u>all</u> three versions)
$\tan(2u)$		Double-Angle ID
$\sin^2(u)$		Power-Reducing ID (PRI)
$\cos^2(u)$		Power-Reducing ID (PRI)
$\sin\left(\frac{\theta}{2}\right)$		Half-Angle ID
$\cos\left(\frac{\theta}{2}\right)$		Half-Angle ID
$\tan\left(\frac{\theta}{2}\right)$		Half-Angle ID (write <u>all</u> three versions)

2) Use the Sum Identities to find the exact values of the following. (A, B)

a) $\sin\left(75^\circ\right)$

b) $\cos\left(75^\circ\right)$

c) $\tan\left(75^\circ\right)$. Remember to rationalize the denominator.

3) Find the exact value of $\sin\left(22.5^\circ\right)$, or $\sin\left(\dfrac{\pi}{8}\right)$. (A, B)

4) Find the exact value of $\cos\left(22.5^\circ\right)$, or $\cos\left(\dfrac{\pi}{8}\right)$. (A, B)

5) Find the exact values of the following expressions. (A, B)

a) $\sin\left(80^\circ\right)\cos\left(20^\circ\right)-\cos\left(80^\circ\right)\sin\left(20^\circ\right)$

b) $\cos\left(55^\circ\right)\cos\left(5^\circ\right)-\sin\left(55\right)^\circ\sin\left(5^\circ\right)$

c) $2\sin\left(\dfrac{3\pi}{8}\right)\cos\left(\dfrac{3\pi}{8}\right)$

d) $2\cos^2\left(\dfrac{\pi}{12}\right)-1$

6) Simplify $\cos^4\left(\theta\right)-\sin^4\left(\theta\right)$. (A, B)

7) Simplify $\dfrac{\sin\left(2x\right)\cos\left(2x\right)}{3\cos^2\left(2x\right)-3\sin^2\left(2x\right)}$. (A, B)

8) Verifying the following identities. (A, B)

a) $\sin\left(\theta+\pi\right)=-\sin\left(\theta\right)$. (Also, try to see why this is true using the Unit Circle.)

b) $\dfrac{\sin\left(2x\right)}{1+\cot^2\left(x\right)}=2\sin^3\left(x\right)\cos\left(x\right)$

c) Verify the Sum Identity for tangent, $\tan\left(u+v\right)=\dfrac{\tan\left(u\right)+\tan\left(v\right)}{1-\tan\left(u\right)\tan\left(v\right)}$, by using the Sum Identities for sine and cosine.

9) For each of the equations below, find all real solutions, and find the particular solutions in the interval $[0, 2\pi)$.

a) $4\sin(x)\cos(x) - 1 = 0$

b) $\cos(2x) + 3\cos(x) + 2 = 0$

c) $\sin(2x) = \sin(x)$

10) Rewrite $\sin[2\arcsin(x)]$ as an equivalent algebraic expression. Assume x is in $[-1, 1]$.

11) Use the Power-Reducing Identities (PRIs) to rewrite the expression $\cos^4 x$ using only the first power of cosine expressions (and no other powers). Fill in the blanks below with real numbers:

$$\cos^4(x) = \boxed{} + \boxed{}\cos(2x) + \boxed{}\cos(4x)$$

12) Rewrite each expression below using either a Product-to-Sum Identity or a Sum-to-Product Identity. Use Even / Odd (Negative-Angle) Identities where appropriate.

a) $\cos(3\theta)\cos(5\theta)$

b) $\cos(5\alpha) + \cos(3\alpha)$

c) $\sin(3x) + \sin(x)$

d) $\sin(9\theta)\cos(10\theta)$

e) $\sin(4x)\sin(x)$

f) $\cos(7x) - \cos(x)$

g) $\sin(8\alpha) - \sin(2\alpha)$

h) $\cos(5\alpha)\sin(4\alpha)$

YOU DO <u>NOT</u> HAVE TO MEMORIZE THE PRODUCT-TO-SUM IDENTITIES; THEY WILL BE PROVIDED TO YOU IF NECESSARY ON THE EXAM. THE SAME GOES FOR THE SUM-TO-PRODUCT IDENTITIES.

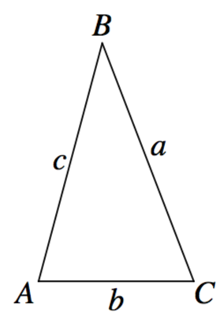
CHAPTER 6:
Additional Topics in Trigonometry

(A) means "refer to Part A," (B) means "refer to Part B," etc.
(Calculator) means "use a calculator." Otherwise, do not use a calculator.

Write units in your final answers where appropriate. Try to avoid rounding intermediate results; if you do round off, do it to at least five significant digits.

SECTION 6.1: THE LAW OF SINES

1) (Given AAS information). Consider the triangle below. The line segment $\overline{AB}$ lies along a riverbank. A canoe leaves Point *A* and takes a straight path along the line segment $\overline{AC}$. A swimmer leaves Point *B* and swims along a straight path represented by the line segment $\overline{BC}$. The canoe and the swimmer meet at Point *C*. We know that the swimmer swims 36.2 meters to meet the canoe. That is, $a = 36.2$ m. Also, Angle *A* has measure 75 degrees, and Angle *B* has measure 36 degrees. (A-C, F) (Calculator)

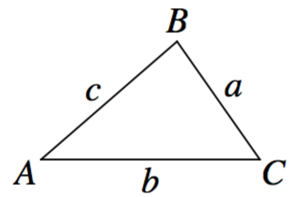


a) Find *c*, the length of $\overline{AB}$. This is the distance between the starting point of the canoe and the starting point of the swimmer along the riverbank. Round off the answer to the nearest tenth of a meter.

b) Find *b*, the length of $\overline{AC}$. This is the distance traveled by the canoe until it meets the swimmer. Round off the answer to the nearest tenth of a meter.

c) Find the area of triangle *ABC*. This is the area of the portion of the river enclosed by the riverbank and the paths of the swimmer and the canoe. Round off the answer to the nearest square meter.

2) (Given ASA information). Consider the triangle below. Soldiers at Point A and Point C fire anti-aircraft missiles simultaneously at an enemy aircraft at Point B. We know that the soldiers are 273 feet apart. That is, $b = 273 \text{ ft}$. The paths of the missiles are represented by line segments $\overline{AB}$ and $\overline{CB}$ below. Angle A has measure 41 degrees, and Angle C has measure 55 degrees. (A-C, F) (Calculator)

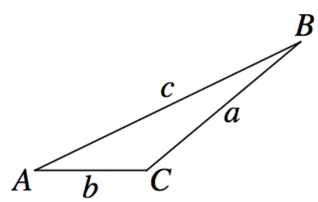


a) Find a, the length of $\overline{CB}$. This is the distance traveled by the missile fired by the soldier at Point C. Round off the answer to the nearest hundredth of a foot.
b) Find c, the length of $\overline{AB}$. This is the distance traveled by the missile fired by the soldier at Point A. Round off the answer to the nearest hundredth of a foot.
c) Find the area of triangle ABC. This is the area of the region in the air enclosed by the missile paths and the strip of ground connecting the soldiers represented by $\overline{AC}$. Round off the answer to the nearest square foot.

(Note: You will not be tested on the "Ambiguous SSA Case" on your exam.)

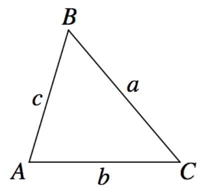
SECTION 6.2: THE LAW OF COSINES

1) (Given SSS information). Consider the triangle below. A hill with a smooth incline of length 52.7 feet is represented by line segment $\overline{CB}$. An observer stands at point A. The observer's shoes are 29.4 feet from the base of the hill and are 77.6 feet from the top of the hill. That is, $a = 52.7 \text{ ft}$, $b = 29.4 \text{ ft}$, and $c = 77.6 \text{ ft}$. (Calculator)



a) Find the measure of Angle A, the angle of elevation from the observer's shoes to the top of the hill. Round off the answer to the nearest tenth of a degree. (A-G)
b) Find the measure of Angle C, the obtuse angle between the ground and the hill. Round off the answer to the nearest tenth of a degree. (A-G)
c) Someone claims that the observer's shoes are actually 9.4 feet from the base of the hill. Could s/he be correct? Why or why not? (D)
d) **ADDITIONAL PROBLEM.** Find the area of triangle ABC using Heron's Formula. Round off the answer to the nearest square foot. (H)

2) (Given SAS information). Consider the triangle below. Two drivers leave their home at Point *B* and drive along straight roads, represented by line segments $\overline{BA}$ and $\overline{BC}$. The angle between the roads is 57°. At noon, one driver has driven 12.6 miles, and the other driver has driven 15.8 miles. In the figure below, $c = 12.6$ mi, and $a = 15.8$ mi. What is the distance between the two cars at noon? (That is, what is *b*, the length of line segment $\overline{AC}$?) Round off the answer to the nearest tenth of a mile. Points *A* and *C* represent the positions of the cars at noon. (Calculator)



SECTION 6.3: VECTORS IN THE PLANE

Direction angles should be given in the interval $\left[0^\circ, 360^\circ\right)$.

1) Consider the vector [representation] that is directed from the point $(3, -7)$ to the point $(5, -4)$ in the usual *xy*-plane, where distances are measured in meters. (A, D, E)
   a) Give the component form $\langle x, y \rangle$ of the vector.
   b) Find the length (or magnitude) of the vector.
   c) Find the direction angle of the vector. Round off the angle measure to the nearest tenth of a degree. (Calculator)
2) Consider the vector [representation] that is directed from the point $(-1, 4)$ to the point $(-6, 1)$ in the usual *xy*-plane, where distances are measured in meters. (A, D, E)
   a) Give the component form $\langle x, y \rangle$ of the vector.
   b) Find the length (or magnitude) of the vector.
   c) Find the direction angle of the vector. Round off the angle measure to the nearest hundredth of a degree. (Calculator)
3) Consider the vectors **v** and **w**, where $\mathbf{v} = \langle 1, 2 \rangle$ and $\mathbf{w} = \langle 3, 1 \rangle$.
   a) Draw **v**, $\frac{1}{3}\mathbf{v}$, and $-3\mathbf{v}$ as position vectors in three different *xy*-planes. Use the same scale on all of the coordinate axes. (A, B, D)
   b) Give the component form $\langle x, y \rangle$ of the resultant vector $\mathbf{v} + \mathbf{w}$. (D, F)
   c) Show how $\mathbf{v} + \mathbf{w}$ can be obtained graphically by using the Triangle Law. (C, D)
   d) Repeat c), but use the Parallelogram Law. (C, D)
   e) Give the component form $\langle x, y \rangle$ of the vector $3\mathbf{w} - 4\mathbf{v}$. (D, F)

4) Find the $\langle x, y\rangle$ component form of the vector **v** that has magnitude 12 feet and direction angle 48° in the usual *xy*-plane. Distances are measured in feet. Round off the *x* and *y* components to the nearest tenth of a foot. (D, E) (Calculator)

5) Let $\mathbf{v} = -2\mathbf{i} + 5\mathbf{j}$. (D-G)

   a) Find the unit vector in the direction of **v**. Write it in $\langle x, y\rangle$ component form. Give an **exact** answer; do **not** approximate. Rationalize denominators.
   b) Find the direction angle of **v**. Round off the angle measure to the nearest tenth of a degree. (Calculator)
   c) Find the vector of magnitude 4 in the direction of **v**. Write the vector in $\langle x, y\rangle$ component form. Give an **exact** answer; do **not** approximate. Rationalize denominators.

6) A vector **v** is represented by a directed arrow in the usual *xy*-plane with initial point $(-4, 2)$ and terminal point $(7, -5)$. (A, D-G)

   a) Find the direction angle of **v**. Round off the angle measure to the nearest hundredth of a degree. (Calculator)
   b) Find the vector of magnitude 20 in the direction of **v**. Write the vector in $\langle x, y\rangle$ component form. Give an **exact** answer; do **not** approximate. Rationalize denominators.

7) Do the vectors $3\mathbf{i} + 4\mathbf{j}$ and $15\mathbf{i} + 20\mathbf{j}$ have the same direction? (A, B, G)

8) Do the vectors $3\mathbf{i} + 4\mathbf{j}$ and $-15\mathbf{i} - 20\mathbf{j}$ have the same direction? (A, B, G)

9) A rock is thrown with an initial velocity of 27.3 miles per hour, at an angle of 42° from the horizontal. (D, E) (Calculator)

   a) What is the rock's horizontal component of initial velocity? Round off to three significant digits.
   b) What is the rock's vertical component of initial velocity? Round off to three significant digits.

SECTION 6.4: VECTORS AND DOT PRODUCTS

1) Let $\mathbf{v} = \langle 2, 4 \rangle$ and $\mathbf{w} = \langle -3, 5 \rangle$. Evaluate the dot product $\mathbf{v} \bullet \mathbf{w}$. (A)

2) Assume that **v** and **w** are vectors in the plane. For each expression below, write "scalar" if it represents a scalar, "vector" if it represents a vector, or "undefined" if it is undefined.
   a) $(\mathbf{v} \bullet \mathbf{w}) \| \mathbf{w} \|$; b) $(\mathbf{v} \bullet \mathbf{w}) \mathbf{w}$; c) $\mathbf{vw}$; d) $(\mathbf{v} + \mathbf{w}) \bullet \mathbf{w}$; e) $(\mathbf{v} + \mathbf{w}) \bullet \| \mathbf{w} \|$
   f) $(\mathbf{v} \bullet \mathbf{w}) \bullet \mathbf{v}$

3) Assume that **v** is a vector in the plane such that $\mathbf{v} \bullet \mathbf{v} = 100$. Find $\| \mathbf{v} \|$. (B)

4) Use the properties of the dot product to prove that, if **v** and **w** are vectors in the plane, then $\| \mathbf{v} + \mathbf{w} \|^2 = \| \mathbf{v} \|^2 + 2(\mathbf{v} \bullet \mathbf{w}) + \| \mathbf{w} \|^2$. (B).

5) Refer to Exercise 4. If **v** and **w** are orthogonal vectors in the plane, then how does the expression $\| \mathbf{v} \|^2 + 2(\mathbf{v} \bullet \mathbf{w}) + \| \mathbf{w} \|^2$ simplify? (C)

6) Refer to Exercise 5. If **v** and **w** are nonzero orthogonal vectors in the plane, then what famous theorem is represented by the statement $\| \mathbf{v} + \mathbf{w} \|^2 = \| \mathbf{v} \|^2 + \| \mathbf{w} \|^2$? Experiment and sketch some vectors to demonstrate the statement.

7) Find the angle between **v** and **w**, where $\mathbf{v} = \langle 4, 1 \rangle$ and $\mathbf{w} = \langle 3, 2 \rangle$, to the nearest tenth of a degree. (First sketch the vectors as position vectors using the same scale for the coordinate axes. Will the angle be acute, right, or obtuse?) (A, C) (Calculator)

8) Find the angle between **v** and **w**, where $\mathbf{v} = \langle -2, 5 \rangle$ and $\mathbf{w} = \langle 1, -6 \rangle$, to the nearest tenth of a degree. (First sketch the vectors as position vectors using the same scale for the coordinate axes. Will the angle be acute, right, or obtuse?) (A, C) (Calculator)

9) A triangle has vertices $A(4, 7)$, $B(1, 2)$, and $C(6, 3)$. Find the measure of Angle ABC to the nearest tenth of a degree. (A, C) (Calculator)

10) For each of a) through c) below, assume that **v** and **w** are nonzero vectors in the plane, and find the angle between **v** and **w**. (C)
   a) Find the angle between **v** and **w**, where $\mathbf{w} = 5\mathbf{v}$.
   b) Find the angle between **v** and **w**, where $\mathbf{w} = -\mathbf{v}$.
   c) Find the angle between **v** and **w**, where $\mathbf{v} \bullet \mathbf{w} = 0$.

11) Are the vectors $\langle 2, 6 \rangle$ and $\langle 3, -1 \rangle$ orthogonal? (A, C)

12) Are the vectors $\langle 3, 7 \rangle$ and $\langle -7, 2 \rangle$ orthogonal? (A, C)

13) What real values for c will make the vectors $\langle c, -1 \rangle$ and $\langle c, c \rangle$ orthogonal? (A, C)

14) Assume that **v** and **w** are nonzero vectors in the plane. The component of **w** along **v** is given by: $\text{comp}_{\mathbf{v}} \mathbf{w} = \|\mathbf{w}\| \cos(\theta)$, where θ is the angle between **v** and **w**.

    Prove that $\text{comp}_{\mathbf{v}} \mathbf{w} = \dfrac{\mathbf{v} \bullet \mathbf{w}}{\|\mathbf{v}\|}$. (A, C, D)

15) Refer to Examples 7 and 14. Let $\mathbf{v} = \langle 4, 1 \rangle$ and $\mathbf{w} = \langle 3, 2 \rangle$, as in Example 7.

    Find $\text{comp}_{\mathbf{v}} \mathbf{w}$. (A, D)

SECTION 6.5: TRIGONOMETRIC (AND EULER / EXPONENTIAL) FORMS OF A COMPLEX NUMBER

(Skip; no homework)