CHAPTER 9: 

Discrete Mathematics

(A) means “refer to Part A,” (B) means “refer to Part B,” etc. Most of these exercises can be done without a calculator, though one may be permitted.

SECTION 9.1: SEQUENCES AND SERIES, and SECTION 9.6: COUNTING PRINCIPLES

1) Let \( a_n = n^2 + n \). Write \( a_1, a_2, \) and \( a_3 \). (A)

2) Let \( a_n = (-1)^n (2n) \). Write \( a_1, a_2, a_3, \) and \( a_4 \). (A-C)

3) Let \( a_n = (-1)^{n-1} (2n - 1) \). Write \( a_1, a_2, a_3, \) and \( a_4 \). (A-C)

4) Evaluate 6!. (D)

5) How many ways are there to order five tasks on a “To Do” list? (D, E)

6) Seven of your friends are sitting in a room. You have two identical plane tickets that you will give to two of them. In how many ways can this be done? (D, E)

7) Ten basketball players are to be divided into two teams of five people each. One team will be called “Team A,” and the other team will be called “Team 1.” In how many ways can the players be assigned to the teams? We do not yet care about positions on the teams. (D, E)

8) Simplify the expressions. Assume that \( n \) is an integer such that \( n \geq 2 \). (D, E)
   a) \((n+1)(n!)\)
   b) \(\frac{(n+2)!}{n!}\)
   c) \(\frac{(n-1)!}{(n+1)!}\)
   d) \(\frac{(3n-2)!}{(3n+3)!}\). (Do not perform any multiplications.)
9) Consider the sequence recursively defined by:
\[
\begin{align*}
    a_1 &= 4 \\
    a_{k+1} &= a_k + 10 \quad (k \in \mathbb{Z}, \ k \geq 1).
\end{align*}
\]

Find \(a_1, a_2, a_3,\) and \(a_4\). This sequence is an arithmetic sequence, which we will discuss further in Section 9.2. (F)

10) Consider the sequence recursively defined by:
\[
\begin{align*}
    a_1 &= 2 \\
    a_{k+1} &= \frac{1}{2} a_k \quad (k \in \mathbb{Z}, \ k \geq 1).
\end{align*}
\]

Find \(a_1, a_2, a_3,\) and \(a_4\). This sequence is a geometric sequence, which we will discuss further in Section 9.3. (F)

11) Consider the sequence recursively defined by:
\[
\begin{align*}
    a_1 &= -1 \\
    a_{k+1} &= 3a_k - 2 \quad (k \in \mathbb{Z}, \ k \geq 1).
\end{align*}
\]

Find \(a_1, a_2, a_3,\) and \(a_4\). (F)

12) Consider the sequence recursively defined by:
\[
\begin{align*}
    a_1 &= 2 \\
    a_2 &= 3 \\
    a_{k+2} &= a_{k+1}a_k \quad (k \in \mathbb{Z}, \ k \geq 1)
\end{align*}
\]

Find \(a_1, a_2, a_3, a_4,\) and \(a_5\). (F)

13) Evaluate \(\sum_{k=1}^{4} k\). (G)

14) Evaluate \(\sum_{j=3}^{7} (j^2 + 1)\). (G)

15) Evaluate \(\sum_{i=2}^{4} \frac{(-1)^i}{i}\). (i is not the imaginary unit here.) (G)

16) Consider the series \(\sum_{k=1}^{\infty} a_k\), where \(a_k = 3^k\). Find \(S_4\), the fourth partial sum of this series. (G, H)
17) Write a nonrecursive expression (formula) for the apparent general \( n^{th} \) term, \( a_n \), for each of the following sequences. Let \( a_1 \) be the initial term; i.e., assume that \( n \) begins with 1. (A-D)

a) \( 7, 8, 9, 10, 11, \ldots \)

b) \( 5, 10, 15, 20, 25, \ldots \)

c) \( 4, 7, 10, 13, 16, \ldots \)

d) \( 1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \frac{1}{120}, \ldots \)

e) \( 7, \frac{7}{4}, \frac{7}{9}, \frac{7}{16}, \frac{7}{25}, \ldots \)

f) \( \frac{2}{3}, -\frac{4}{5}, \frac{6}{7}, -\frac{8}{9}, \frac{10}{11}, \ldots \)

g) \(-2, 4, -8, 16, -32, \ldots \)

18) Express the apparent series using summation notation. Use \( k \) as the index of summation. (A-D, G, H)

a) \( 3 + 6 + 9 + 12 + 15 + 18 \); this is a finite series.

b) \( \frac{1}{4} - \frac{1}{16} + \frac{1}{64} - \frac{1}{256} + \frac{1}{1024} - \ldots \); this is an infinite series.
SECTION 9.2: ARITHMETIC SEQUENCES and PARTIAL SUMS

1) Consider the arithmetic sequence: \(-5, 1, 7, 13, 19, \ldots\) (A)
   a) What is the initial term, \(a\)?
   b) What is the common difference, \(d\)?

2) Consider the arithmetic sequence: \(\frac{3}{2}, 1, \frac{1}{2}, 0, -\frac{1}{2}, \ldots\) (A)
   a) What is the initial term, \(a\)?
   b) What is the common difference, \(d\)?

3) Consider the arithmetic sequence with initial term 7 and common difference 3. Assume that the initial term is \(a_1\). (A, B)
   a) Write the first four terms of this sequence.
   b) Find \(S_4\), the fourth partial sum.
   c) Find \(a_{60}\), the 60th term of this sequence.

4) Consider the arithmetic sequence: \(2, -3, -8, -13, -18, \ldots\) (A, B)
   a) Write a simplified, nonrecursive expression (formula) for the general \(n^{th}\) term, \(a_n\), for this sequence. Let \(a_1\) be the initial term.
   b) Use a) to find \(a_{387}\). (Calculator)

5) An arithmetic sequence has \(a_1 = 6\) and \(a_{200} = 1399\). Find \(a_{123}\). (Calculator)
SECTION 9.3: GEOMETRIC SEQUENCES, PARTIAL SUMS, and SERIES

1) Consider the geometric sequence: 4, 20, 100, 500, 2500, …. (A)
   a) What is the initial term, \(a\)?
   b) What is the common ratio, \(r\)?

2) Consider the geometric sequence: \(\frac{6}{7}, -\frac{2}{7}, \frac{2}{21}, -\frac{2}{63}, \frac{2}{189}\) …. (A)
   a) What is the initial term, \(a\)?
   b) What is the common ratio, \(r\)?

3) Consider the geometric sequence with initial term 5 and common ratio \(-4\). Assume that the initial term is \(a_1\). (A, B)
   a) Write the first four terms of this sequence.
   b) Find \(S_4\), the fourth partial sum.
   c) Find \(a_{12}\), the 12\(^{th}\) term of this sequence. (Calculator)
   d) As \(n \to \infty\), do the terms of the sequence approach a real number? If so, what number?

4) Consider the geometric sequence: \(\frac{2}{9}, \frac{1}{3}, \frac{1}{2}, \frac{3}{4}, \frac{9}{8}\) …. (A, B)
   a) Write a simplified, nonrecursive expression (formula) for the general \(n^{th}\) term, \(a_n\), for this sequence. Let \(a_1\) be the initial term.
   b) Use a) to find \(a_{10}\). (Calculator)
   c) Verify your answer to b) by recursively using the common ratio to find \(a_6, a_7, a_8, a_9, a_{10}\).
   d) As \(n \to \infty\), do the terms of the sequence approach a real number? If so, what number?
5) A geometric sequence has \( a_1 = 3 \) and \( a_4 = -\frac{24}{125} \). (A, B)

   a) Find \( a_2 \) and \( a_3 \).

   b) As \( n \to \infty \), do the terms of the sequence approach a real number? If so, what number?

6) Let \( \{a_n\} \) be the sequence from Exercise 3. Does the series \( \sum_{n=1}^{\infty} a_n \) converge or diverge? If the series converges, find its sum. (D)

7) Let \( \{a_n\} \) be the sequence from Exercise 4. Does the series \( \sum_{n=1}^{\infty} a_n \) converge or diverge? If the series converges, find its sum. (D)

8) Let \( \{a_n\} \) be the sequence from Exercise 5. Does the series \( \sum_{n=1}^{\infty} a_n \) converge or diverge? If the series converges, find its sum. (D)

9) For what values of \( x \) does the series \( \sum_{n=1}^{\infty} (3x)^n \) converge? (D)

10) Consider the infinite geometric series: \( \frac{2}{3} + 2 + 6 + 18 + 54 + \ldots \). (D)

    a) Write the series using summation notation.

    b) Is the series convergent or divergent?

    c) If the series is convergent, find its sum. If the series is divergent, write “No sum.”

11) Consider the infinite geometric series: \( -3 + 3 - 3 + 3 - 3 + \ldots \). (D)

    a) Write the series using summation notation.

    b) Is the series convergent or divergent?

    c) If the series is convergent, find its sum. If the series is divergent, write “No sum.”
12) Consider the infinite geometric series: \( \frac{5}{4} + \frac{5}{16} - \frac{5}{64} + \frac{5}{256} - \cdots \) (D)

a) Write the series using summation notation.

b) Is the series convergent or divergent?

c) If the series is convergent, find its sum. If the series is divergent, write “No sum.”

13) Write 0.39 as a nice (simplified) fraction of the form \( \frac{\text{integer}}{\text{integer}} \). (D)

14) ADDITIONAL PROBLEM: Write 0.5172 as a nice (simplified) fraction of the form \( \frac{\text{integer}}{\text{integer}} \). (D)

SECTION 9.4: MATHEMATICAL INDUCTION

1) Prove using mathematical induction: \( 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2} \quad (\forall n \in \mathbb{Z}^+) \),

where \( \mathbb{Z}^+ \) is the set of positive integers.

2) Use the statement proven in 1) to evaluate \( \sum_{k=1}^{1000} k \).

SECTION 9.5: THE BINOMIAL THEOREM

1) Expand and simplify \( (x + y)^5 \) using the Binomial Theorem. (A, B)

2) Expand and simplify \( (a + b)^6 \) using the Binomial Theorem. (A, B)

3) Expand and simplify \( (2x + 3y)^3 \) using the Binomial Theorem. (A-C)

4) Expand and simplify \( (a - 2b)^4 \) using the Binomial Theorem. (A-C)
CHAPTER 9:

Discrete Mathematics

SECTION 9.1: SEQUENCES AND SERIES, and
SECTION 9.6: COUNTING PRINCIPLES

1) \( a_1 = 2, a_2 = 6, a_3 = 12 \)

2) \( a_1 = -2, a_2 = 4, a_3 = -6, a_4 = 8 \)

3) \( a_1 = 1, a_2 = -3, a_3 = 5, a_4 = -7 \)

4) 720

5) 120 ways

6) 21 ways

7) 252 ways

8)
   a) \( (n+1)! \)
   
   b) \( (n+2)(n+1), \) or \( n^2 + 3n + 2 \)
   
   c) \( \frac{1}{n(n+1)}, \) or \( \frac{1}{n^2 + n} \)
   
   d) \( \frac{1}{(3n+3)(3n+2)(3n+1)(3n)(3n-1)} \)

9) \( a_1 = 4, a_2 = 14, a_3 = 24, a_4 = 34 \)

10) \( a_1 = 2, a_2 = 1, a_3 = \frac{1}{2}, a_4 = \frac{1}{4} \)

11) \( a_1 = -1, a_2 = -5, a_3 = -17, a_4 = -53 \)

12) \( a_1 = 2, a_2 = 3, a_3 = 6, a_4 = 18, a_5 = 108 \)
13) 10
14) 140
15) \(\frac{5}{12}\)
16) 120

17)  
   a) \(a_n = n + 6\)
   b) \(a_n = 5n\)
   c) \(a_n = 3n + 1\)
   d) \(a_n = \frac{1}{n!}\)
   e) \(a_n = \frac{7}{n^2}\)
   f) \(a_n = (-1)^{n-1} \frac{2n}{2n+1}, \text{ or } a_n = (-1)^{n+1} \frac{2n}{2n+1}\)
   g) \(a_n = (-1)^n 2^n, \text{ or } a_n = (-2)^n\)

18)  
   a) \(\sum_{k=1}^{6} 3k\)
   b) \(\sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{4^k}, \text{ or } \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{4^k}, \text{ or } \sum_{k=1}^{\infty} \left( -\frac{1}{4} \right)^k\)
SECTION 9.2: ARITHMETIC SEQUENCES and PARTIAL SUMS

1) a) \(-5\); b) 6

2) a) \(\frac{3}{2}\); b) \(-\frac{1}{2}\)

3) a) 7, 10, 13, 16; b) 46; c) 184

4) a) \(a_n = 7 - 5n\), which is simplified from \(a_n = 2 + (n-1)(-5)\)
   b) \(-1928\)

5) 860. (Hint: First find \(d\). \(d = 7\).)

SECTION 9.3: GEOMETRIC SEQUENCES, PARTIAL SUMS, and SERIES

1) a) 4; b) 5

2) a) \(\frac{6}{7}\); b) \(-\frac{1}{3}\)

3) a) 5, -20, 80, -320; b) -255; c) -20,971,520; d) No (What is \(r\)?)

4) a) \(a_n = \left(\frac{2}{9}\right)\left(\frac{3}{2}\right)^{n-1}\), or \(a_n = \frac{3^{n-3}}{2^{n-2}}\)
   b) \(\frac{2187}{256}\)
   c) \(a_6 = \frac{27}{16}\), \(a_7 = \frac{81}{32}\), \(a_8 = \frac{243}{64}\), \(a_9 = \frac{729}{128}\), \(a_{10} = \frac{2187}{256}\)
   d) No (What is \(r\)?)
5) 
   a) \(a_2 = -\frac{6}{5}, \ a_3 = \frac{12}{25}\). (Hint: First find \(r\). \(r = -\frac{2}{5}\).) 
   
   b) Yes; 0 (What is \(r\)?) 

6) The series diverges. (What is \(r\)?) 

7) The series diverges. (What is \(r\)?) 

8) The series converges. (What is \(r\)?) The sum is \(\frac{15}{7}\). 

9) \( \left\{ x \in \mathbb{R} \left| -\frac{1}{3} < x < \frac{1}{3} \right. \right\}, \) or the interval \( \left( -\frac{1}{3}, \frac{1}{3} \right) \) 

10) 
   a) \(\sum_{n=1}^{\infty} 2(3^{n-2})\), which is simplified from \(\sum_{n=1}^{\infty} \frac{2}{3}(3^{n-1})\). It can be rewritten as \(\sum_{i=-1}^{\infty} 2(3^i)\). 
      (The index of summation could be something other than \(n\) or \(i\).) 
   
   b) Divergent 
   
   c) No sum 

11) 
   a) \(\sum_{n=1}^{\infty} 3(-1)^n\), which is simplified from \(\sum_{n=1}^{\infty} -3(-1)^{n-1}\). 
      (The index of summation could be something other than \(n\) or \(i\).) 
   
   b) Divergent 
   
   c) No sum 

12) 
   a) \(\sum_{n=1}^{\infty} 5\left(-\frac{1}{4}\right)^{n-1}\), which can be rewritten as \(\sum_{i=0}^{\infty} 5\left(-\frac{1}{4}\right)^i\). 
      (The index of summation could be something other than \(n\) or \(i\).) 
   
   b) Convergent 
   
   c) 4
13) \[ \frac{13}{33} \]

14) \[ \frac{5167}{9990} \]. Hint: Rewrite \(0.5172\) as \(0.5 + 0.0172\).

**SECTION 9.4: MATHEMATICAL INDUCTION**

1) See the notes.

2) 500,500

**SECTION 9.5: THE BINOMIAL THEOREM**

1) \( x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5 \)

2) \( a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6 \)

3) \( 8x^3 + 36x^2y + 54xy^2 + 27y^3 \)

4) \( a^4 - 8a^3b + 24a^2b^2 - 32ab^3 + 16b^4 \)