QUIZ 1A - SOLUTIONS

(CHAPTER 0: PRELIMINARY TOPICS)
MATH 141 – SPRING 2025 – KUNIYUKI
90 POINTS TOTAL

SHORTER PROBLEMS (33 POINTS)

1) (1 point). The symbol $\forall$ means which of the following? (Box in one.)

   [For all]   There exists   Is a member of

2) (6 points total).

   a) Write the **converse** of this statement:

   "If it rains, then I bring an umbrella."

   ["If I bring an umbrella, then it rains."]

   b) Write the **contrapositive** of this statement:

   "If it rains, then I bring an umbrella."

   ["If I don't bring an umbrella, then it doesn't rain."]

   c) Which is logically equivalent to the given statement? (Box in one.)

   Its converse   Its inverse   **[Its contrapositive]**

3) (3 points). Give the piecewise definition of $|a|$ $\left(\text{where } a \in \mathbb{R}\right)$ given in class.

$$\boxed{|a| = \begin{cases} a, & \text{if } a \geq 0 \\ -a, & \text{if } a < 0 \end{cases}}$$

4) (2 points). Mathematically express the following as an absolute value inequality: The distance between x and 5 on the real number line is less than 7.

$$\boxed{|x-5| < 7, \text{ or } |5-x| < 7}$$

5) (4 points). Solve the correct absolute value inequality from Problem 4); that is, solve the correct answer to Problem 4). Write the solution set in interval form (the form with parentheses and/or brackets).

$$|x-5| < 7$$

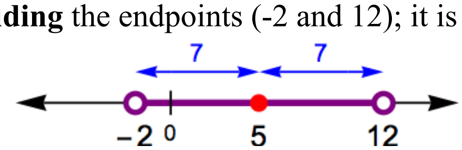
$$-7 < x-5 < 7$$

Add 5 to all three parts of this compound inequality.

$$-2 < x < 12$$

The solution set in interval form is: $\boxed{(-2,12)}$.

Note: See Problem 4). This is the set of numbers that are **strictly** within 7 units of 5 on the real number line, **excluding** the endpoints (-2 and 12); it is an **open** interval.



6) (1 point). A student says that the expression $(3a+3b)^2$ is equivalent to $3(a+b)^2$. Is the student correct? Box in one: Yes **[No]**

- A **counterexample** is provided by the case $a = 1$, $b = 0$.
- It is **not** generally the case that you can "factor out of a power," unless the exponent is 1 or is shared by the factor; it is true that: $(3a+3b)^2 = [3(a+b)]^2 = 3^2(a+b)^2$.

7) (6 points total). Fill in the boxes with simplified real numbers to make the statements correct.

a) $\frac{9x^2}{16} + 25y^2 = \frac{x^2}{\boxed{\frac{16}{9}}} + \frac{y^2}{\boxed{\frac{1}{25}}}$

Multiplying by a nonzero number is equivalent to **dividing by its reciprocal**.

b) $\frac{3-5x}{x^4} = 3x^{\boxed{-4}} - 5x^{\boxed{-3}} \quad (x \neq 0)$

$$\frac{3-5x}{x^4} = \frac{3}{x^4} - \frac{5x}{x^4} = \frac{3}{x^4} - \frac{5}{x^3} = 3x^{-4} - 5x^{-3}$$

8) (2 points). What is the slope of any line in the *xy*-plane that is perpendicular to the line $y = -4x + 9$?

Answer: $\boxed{\frac{1}{4}}$. The slope of the **given** line is -4. The **opposite reciprocal** of -4 is $\frac{1}{4}$, which is the slope of any line **perpendicular** to the given line.

9) (8 points total; 2 points each). Write the formulas for the following.

Description	Formula	Comments
The **lateral surface area** of a right circular **cylinder** with base radius r and height h	$\text{LSA} = 2\pi rh$	Think: (Circumference of circular base) times (height)
The **volume** of a right circular **cylinder** with base radius r and height h	$V = \pi r^2 h$	Think: (Area of circular base) times (height)
The **volume** of a right circular **cone** with base radius r and height h	$V = \frac{1}{3}\pi r^2 h$	Think: (One-third) the volume of the corresponding (circumscribing) cylinder
The **surface area** of a **sphere** of radius r	$\text{SA} = 4\pi r^2$	

LONGER PROBLEMS (57 POINTS)

10) Simplify $\dfrac{\left(-3x^2\right)^3}{x^{7/3}\cdot\sqrt[3]{x^2}}$ completely. (6 points)

$$\frac{\left(-3x^2\right)^3}{x^{7/3}\cdot\sqrt[3]{x^2}} = \frac{(-3)^3\left(x^2\right)^3}{x^{7/3}\cdot x^{2/3}} = \frac{-27x^6}{x^{\left(\frac{7}{3}+\frac{2}{3}\right)}} = \frac{-27x^6}{x^3} = -27x^{6-3}\ (x\neq 0) = \boxed{-27x^3\ (x\neq 0)}$$

11) (6 points). Factor completely over $\mathbb{Z}$ (that is, using only integer coefficients), as in class, and write as a fraction with no negative exponents:

$3x^{-2}+x^{-3}-2x^{-4}$

- **Factor** out the power of x with the **least exponent**; here, it is x^{-4}.

$$3x^{-2}+x^{-3}-2x^{-4} = x^{-4}\left(3x^{-2-(-4)}+x^{-3-(-4)}-2\right) = x^{-4}\left(3x^2+x-2\right)$$

$$= x^{-4}(3x-2)(x+1) = \boxed{\frac{(3x-2)(x+1)}{x^4}}$$

- How to **factor the "QT," the quadratic trinomial** $\left(3x^2+x-2\right)$ as $(3x-2)(x+1)$:
  - There's only one way to break up $3x^2$ here: $3x$ and x, or vice-versa.
  - The **constant term** of the given trinomial, -2, is **negative**, so the constant terms of the factors must have **opposite** signs.
  - The "Outer" product + the "Inner" product $= 3x-2x = x$, the **linear** term.

12) (5 points). Simplify completely: $\dfrac{2-x}{x^3-8}$

Template for **factoring a Difference of Two Cubes**: $A^3-B^3=(A-B)\left(A^2+AB+B^2\right)$

- The three visible **signs** on the right-hand side follow the **SOAP** pattern: "**S**ame" (as on the left side), "**O**pposite," and "**A**lways **P**lus (+)."
- Here, for x^3-8, $A=x$ and $B=2$: $x^3-8=(x-2)\left(x^2+2x+4\right)$
- The **trinomial** factor, $\left(x^2+2x+4\right)$, is **prime** over $\mathbb{Z}$ by the Test for Factorability.

GCF = 1, and the **discriminant** $b^2-4ac=-12$, which is **not** a perfect square.

$$\frac{2-x}{x^3-8} = \frac{2-x}{(x-2)\left(x^2+2x+4\right)}$$

$$= \frac{\overset{(-1)}{\cancel{(2-x)}}}{\underbrace{\cancel{(x-2)}}_{(1)}\left(x^2+2x+4\right)} \quad \left[\text{Warning: } (2-x)=-(x-2)\right]$$

$$= \boxed{-\frac{1}{x^2+2x+4},\ (x\neq 2)}$$

13) A spherical balloon has volume 300 cubic inches. Find the radius r of this balloon. Write an exact answer, and include appropriate units. (You do not have to rationalize denominators.) Also write an approximate answer in decimal form by rounding off to four significant digits. (6 points)

Let V = the volume of the spherical balloon. Then, $V=\frac{4}{3}\pi r^3$. Solve for r:

$$V=\frac{4}{3}\pi r^3 \quad \Rightarrow \quad (300)=\frac{4}{3}\pi r^3$$

$$\left(\frac{3}{4}\right)(300)=\overset{(1)}{\cancel{\left(\frac{3}{4}\right)\left(\frac{4}{3}\right)}}\pi r^3$$

$$225=\pi r^3$$

$$r^3=\frac{225}{\pi}$$

The radius of the balloon, $r=\sqrt[3]{\frac{225}{\pi}}$ in (See Note.) $\approx$ 4.153 in (See Warning.)

Note: π is an irrational denominator, but there's not much we can do about that!

Warning: To approximate $\sqrt[3]{\frac{225}{\pi}}$ using an older calculator, you should enter: "$225\div\pi=$" **before** taking the cube root.

14) d is directly proportional to the cube of x and inversely proportional to t. Find the **particular** model equation related to this statement if d is 11 when x is 2 and t is 3, as in class. Make sure your model is in simplified form. (By "particular," we mean determine the constant of proportionality.) (6 points)

General model: $d=\frac{kx^3}{t}$; k is a nonzero constant of proportionality or variation $\Rightarrow$

Find k using the given data:

$$11=\frac{k(2)^3}{3}$$

$$11=\frac{8k}{3}$$

$$33=8k$$

$$k=\frac{33}{8} \quad \Rightarrow$$

Substitute this value of k into our general model to obtain our particular model:

$$d=\frac{\left(\frac{33}{8}\right)x^3}{t}$$

$$d=\frac{\left(\frac{33}{\cancel{8}_{(1)}}\right)x^3}{t}\cdot\frac{\overset{(1)}{\cancel{8}}}{8}$$

$$\boxed{d=\frac{33x^3}{8t}}$$

15) For parts a), b), and c) below, consider the points $P(2,-1)$ and $Q(6,2)$ in the usual *xy*-plane. Write all numerical constants in simplest form. Distance is measured in meters. (19 points total)

a) Find the distance between the two points (that is, the length of the line segment $\overline{PQ}$). (5 points)

$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}=\sqrt{(6-2)^2+(2-(-1))^2}=\sqrt{(4)^2+(3)^2}=\sqrt{25}=\boxed{5\text{ m}}$$

b) Find the standard form of the equation of the circle that has $P(2,-1)$ as its center and that passes through the point $Q(6,2)$ as a solution point. Hint: Part a) will help. (5 points)

The **distance** between the two points (our answer to Part a)) is the **radius** of the desired circle. The equation of the circle:

$$(x-h)^2+(y-k)^2=r^2$$

$$(x-2)^2+(y-(-1))^2=(5)^2,\ \text{or}\ \boxed{(x-2)^2+(y+1)^2=25}$$

c) Find the Slope-Intercept Form of the equation of the line $\overleftrightarrow{PQ}$ that passes through the two points *P* and *Q*. Hint: This part can be done without parts a) and b). (9 points)

Find the slope of the line: $m=\frac{\Delta y}{\Delta x}=\frac{y_2-y_1}{x_2-x_1}=\frac{2-(-1)}{6-2}=\frac{3}{4}$

We may use either of the two given points. Let's use $Q(6,2)$.

Method 1: Start w/**Point-Slope Form**.

$$y-y_1=m(x-x_1)\ \Rightarrow$$

$$y-2=\frac{3}{4}(x-6)$$

$$y-2=\frac{3}{4}x-\frac{18}{4}$$

$$y=\frac{3}{4}x-\frac{9}{2}+2$$

$$y=\frac{3}{4}x-\frac{9}{2}+\frac{4}{2}$$

Method 2: Plug (substitute) into **Slope-Intercept Form** directly.

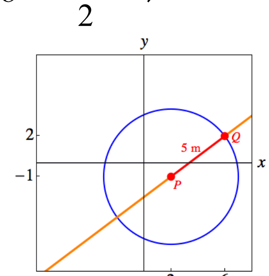
$$y=mx+b\ \Rightarrow$$

$$(2)=\left(\frac{3}{4}\right)(6)+b$$

$$2=\left(\frac{3}{\cancel{4}_{(2)}}\right)\cancel{(6)}^{(3)}+b$$

$$2=\frac{9}{2}+b$$

$$b=-\frac{5}{2}\ \Rightarrow$$



$$\boxed{y=\frac{3}{4}x-\frac{5}{2}}$$

16) Simplify the following expression completely, as in class.
Your final answer must be a single non-compound fraction with no nonpositive exponents. (You do not have to rationalize denominators.)
You may ignore domain issues here. (9 points)

$$\frac{x^4\left(\frac{1}{2}\right)(2x+5)^{-1/2}(2) - (2x+5)^{1/2}(4x^3)}{(x^4)^2}$$

<u>"Clean-up" simplification</u>: (The substitution $u = (2x+5)$ may help.)

$$= \frac{x^4\left(\frac{1}{2}\right)u^{-1/2}(2) - u^{1/2}(4x^3)}{(x^4)^2} = \frac{x^4u^{-1/2} - u^{1/2}(4x^3)}{x^8}$$

<u>Method 1</u>: Factor the numerator.

$$= \frac{x^3u^{-1/2}\left[x - u^{\overbrace{\frac{1}{2}-\left(-\frac{1}{2}\right)}^{=1}}(4)\right]}{x^8} = \frac{\overset{(1)}{x^3}u^{-1/2}[x - 4u]}{\underset{(x^5)}{x^8}}$$

$$= \frac{u^{-1/2}[x-4u]}{x^5}\quad\left(\text{Note: } u^{-1/2} = \frac{1}{u^{1/2}}\right) = \frac{x-4u}{x^5u^{1/2}}$$

$$= \frac{x-4(2x+5)}{x^5(2x+5)^{1/2}} = \frac{x-8x-20}{x^5(2x+5)^{1/2}} = \boxed{\frac{-7x-20}{x^5(2x+5)^{1/2}},\ \text{or}\ -\frac{7x+20}{(x^5)\sqrt{2x+5}}}$$

<u>Method 2</u>: Rewrite as a compound (or complex) fraction.

$$= \frac{\frac{x^4}{u^{1/2}} - u^{1/2}(4x^3)}{x^8} \quad \text{or} \quad \frac{\frac{x^4}{\sqrt{u}} - (4x^3)\sqrt{u}}{x^8}$$

$$= \frac{\left[\frac{x^4}{u^{1/2}} - u^{1/2}(4x^3)\right]}{x^8}\cdot\frac{u^{1/2}}{u^{1/2}} \quad \text{or} \quad \frac{\left[\frac{x^4}{\sqrt{u}} - (4x^3)\sqrt{u}\right]}{x^8}\cdot\frac{\sqrt{u}}{\sqrt{u}}$$

Use the Distributive Property before canceling to obtain the numerator.

$$= \frac{x^4 - (4x^3)u}{x^8u^{1/2}} = \frac{x^3[x-4u]}{x^8u^{1/2}} = \frac{\overset{(1)}{x^3}[x-4u]}{\underset{(x^5)}{x^8}u^{1/2}} = \frac{x-4u}{x^5u^{1/2}} = \frac{x-4(2x+5)}{x^5(2x+5)^{1/2}}$$

$$= \frac{x-8x-20}{x^5(2x+5)^{1/2}} = \boxed{\frac{-7x-20}{x^5(2x+5)^{1/2}},\ \text{or}\ -\frac{7x+20}{(x^5)\sqrt{2x+5}}}$$