QUIZ 1B - SOLUTIONS

(CHAPTER 1: FUNCTIONS)
MATH 141 – SPRING 2025 – KUNIYUKI
60 POINTS TOTAL
No notes or books allowed. A scientific calculator is allowed.

SHORTER PROBLEMS (29 POINTS)

1) (6 points). Write the domain of *f*, where $f(x) = \dfrac{\sqrt{x+2} - \sqrt[3]{x}}{x-3}$, using interval form (the form using parentheses and/or brackets).

• The domain of $\sqrt[3]{x}$ is unrestricted over $\mathbb{R}$. $\sqrt[3]{x}$ is real everywhere on $\mathbb{R}$.

• $\sqrt{x+2}$ is real $\Leftrightarrow x+2 \geq 0 \Leftrightarrow x \geq -2$

• $x - 3 \neq 0 \Leftrightarrow x \neq 3$, so 3 is the only exclusion from the domain based on the denominator, which is not allowed to be 0.

Here is a graph of the domain:

In interval form, the domain is: $\boxed{[-2, 3) \cup (3, \infty)}$.

2) (2 points). Find the *x*-intercept of the graph of $y = \dfrac{x-6}{x+7}$.

We set $y = 0$ and solve for *x*: $0 = \dfrac{x-6}{x+7} \Leftrightarrow x - 6 = 0 \; (x \neq -7) \Leftrightarrow x = 6$.

The only *x*-intercept is at $\boxed{(6, 0)}$.

3) (1 point). Evaluate $[\![7.9]\!]$. (This is the same as $\lfloor 7.9 \rfloor$.)

The greatest integer (or floor) function has the effect of **rounding down** arguments. $[\![7.9]\!] = \boxed{7}$. Note that 7 is the **greatest integer that does not exceed** 7.9.

4) (1 point). The graph of $y = x^6 - 3x^2 + 5$ is symmetric about the… (Box in one:)

x-axis $\qquad$ $\boxed{y\text{-axis}}$ $\qquad$ origin $\qquad$ (none of these)

Let $f(x) = x^6 - 3x^2 + 5$, a polynomial in descending powers of *x* where all exponents on *x* are **even**; the constant term 5 may be seen as $5x^0$. f is an **even** function (and the graph is symmetric about the ***y*-axis**). Proof: $\text{Dom}(f) = \mathbb{R}$, which is **symmetric about 0**.

$$\forall x \in \mathbb{R},\; f(-x) = (-x)^6 - 3(-x)^2 + 5 = x^6 - 3x^2 + 5 = f(x).$$

5) (1 point). The graph of $y = \sqrt[3]{x} + \frac{1}{x}$ is symmetric about the … (Box in one:)

x-axis　　y-axis　　$\boxed{\text{origin}}$　　(none of these)

Let $f(x) = \sqrt[3]{x} + \frac{1}{x}$. Then, f is an **odd** function (and the graph is symmetric about the **origin**), because $\text{Dom}(f) = \mathbb{R} \setminus \{0\}$, which is **symmetric about 0**, and $\forall x \in \text{Dom}(f)$,

$$f(-x) = \sqrt[3]{(-x)} + \frac{1}{(-x)} = -\sqrt[3]{x} - \frac{1}{x} = -f(x).$$

6) (6 points total). If the point $(5, 2)$ lies on the graph of $y = f(x)$, where f is a one-to-one function, what point must then lie on the graph of …

a) … $y = f(x+3) - 1$?

$\boxed{(2,1)}$; the point $(5, 2)$ shifts 3 units **left** and 1 unit **down**.

b) … $y = -f(x)$?

$\boxed{(5,-2)}$, the **reflection** of $(5, 2)$ about the ***x*-axis**.

c) … $y = f^{-1}(x)$?

$\boxed{(2,5)}$. **Reflect** $(5, 2)$ about the line $\boldsymbol{y = x}$ by **switching** x- and y-coordinates.

7) (2 points). Find functions g and f such that $(f \circ g)(x) = \sqrt{x^4 + 5}$. You may <u>not</u> use the identity function. Fill in the blanks:

$\boxed{g(x) = x^4 + 5, \quad f(u) = \sqrt{u}}$. There are many other possibilities.

8) (2 points). Let $f(x) = (7x)^5$. What is $f^{-1}(x)$?

<u>Answer</u>:

$$f^{-1}(x) = \boxed{\frac{\sqrt[5]{x}}{7}}.$$

<u>Conceptual Approach</u>. f multiplies its input by 7 and raises the result to the 5^{th} power.
f^{-1} takes the 5^{th} root of its input and divides the result by 7 to undo f.
Think: Invert the steps in reverse order.

Mechanical Approach.

Step 1: Replace $f(x)$ with y.

$$y = (7x)^5$$

Step 2: Switch x and y.

$$x = (7y)^5$$

Step 3: Solve for y.

$$\sqrt[5]{x} = 7y$$

$$\frac{\sqrt[5]{x}}{7} = y$$

$$y = \frac{\sqrt[5]{x}}{7}$$

Step 4: Replace y with $f^{-1}(x)$

$$f^{-1}(x) = \boxed{\frac{\sqrt[5]{x}}{7}}$$

9) (6 points). Match the equations with their corresponding graphs by writing the appropriate letters in the blanks. The x- and y-axes are not necessarily scaled the same way.

The graph of $y = \sqrt[3]{x}$ is Graph __**B**__. (Lazy snake / "S" graph)

The graph of $y = x^{2/3}$ is Graph __**C**__. (Bird beak / Tornado / Funnel)

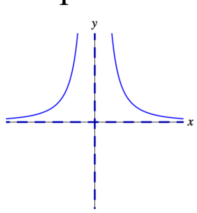
The graph of $y = \frac{1}{x}$ is Graph __**F**__. (Hyperbola)

The graph of $y = \frac{1}{x^2}$ is Graph __**A**__. (Volcano)

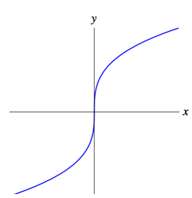
The graph of $y = |x|$ is Graph __**D**__. ("V" graph)

The graph of $y = \sqrt{49 - x^2}$ is Graph __**E**__. (Upper semicircle)

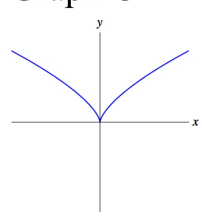
Graph A



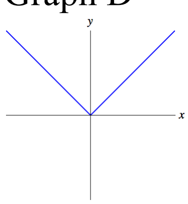
Graph B



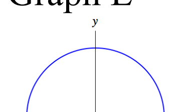
Graph C



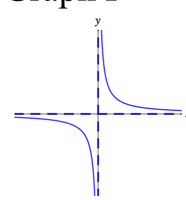
Graph D



Graph E

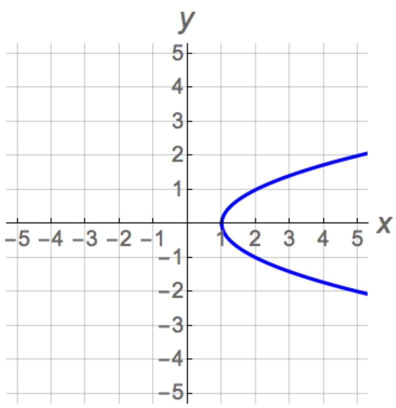


Graph F



- The **domains** of the functions corresponding to A, B, C, D, E, and F are, respectively, $(-\infty, 0) \cup (0, \infty)$, $\mathbb{R}$, $\mathbb{R}$, $\mathbb{R}$, $[-7, 7]$, and $(-\infty, 0) \cup (0, \infty)$.
- B and F are graphs of **odd** functions, while the others are graphs of **even** functions.

10) (2 points). Graph $x = y^2 + 1$ on the grid below.



The graph of $x = y^2$, a **rightward-opening parabola** with vertex at the origin, is translated (shifted) one unit to the **right**.

LONGER PROBLEMS (31 POINTS)

11) f is the function defined piecewise by: $f(x) = \begin{cases} |x| + 1, & -3 \le x < 1 \\ \sqrt{x}, & 1 < x < 4 \end{cases}$

(15 points)

a) Evaluate $f(0)$. (1 point)

$-3 \le 0 < 1$, so use the **top** rule, $f(x) = |x| + 1$. $f(0) = |0| + 1 = 0 + 1 = \boxed{1}$.

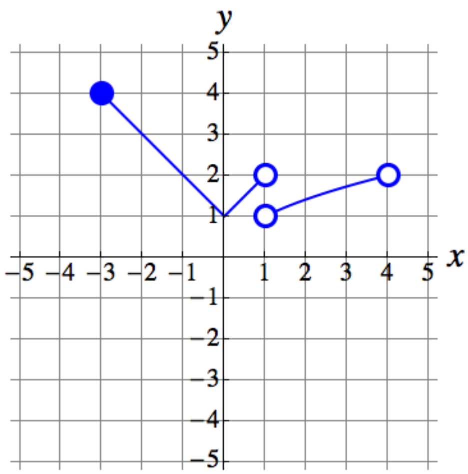
b) Evaluate $f(1)$. (1 point)

$1 \notin \text{Dom}(f)$, so $f(1)$ is $\boxed{\text{undefined}}$.

c) Evaluate $f(2)$. (1 point)

$1 < 2 < 4$, so use the **bottom** rule, $f(x) = \sqrt{x}$. $f(2) = \boxed{\sqrt{2}}$.

d) Graph $y = f(x)$ on the grid below. Be accurate. Clearly indicate whether endpoints are included or excluded, as in class. (8 points)



• The "V" graph of $y = |x| + 1$ has a corner at $(0, 1)$. In Part a), we saw that $f(0) = 1$, confirming that $(0, 1)$ is a point on the graph. The graph is translated **one unit upward** from the parent graph of $y = |x|$. We take a piece of the translated graph.

• The graph of $y = \sqrt{x}$ is a downward curving (concave down) "hook." We take a piece of this graph. In Part c), we saw that $f(2) = \sqrt{2}$, confirming that $(2, \sqrt{2})$ is a point on the graph.

• Locate **endpoints** on the graph of $y = f(x)$ (corresponding to endpoints of the subdomains) and determine if they are **included** in (or **excluded** from) the graph.

Rule	Left endpoints	Right endpoints	Shapes
$\lvert x\rvert + 1$	$x = -3$ $y = \lvert -3\rvert + 1 = 3 + 1 = 4$ **Left endpoint:** $(-3, 4)$. It is **included** because of "$\leq$" (weak inequality).	$x = 1$ $y = \lvert 1\rvert + 1 = 1 + 1 = 2$ **Right endpoint:** $(1, 2)$. It is **excluded** because of "$<$" (strict inequality).	(See above.)
$\sqrt{x}$	$x = 1$ $y = \sqrt{1} = 1$ **Left endpoint:** $(1, 1)$. It is **excluded** because of "$<$" (strict inequality).	$x = 4$ $y = \sqrt{4} = 2$ **Right endpoint:** $(4, 2)$. It is **excluded** because of "$<$" (strict inequality).	(See above.)

e) Give the **domain** of f using interval form (the form with parentheses and/or brackets). (2 points)

$\text{Dom}(f) = \boxed{[-3,1)\cup(1,4)}$, which gives the ***x*-coordinates** "picked up" by the graph of $y = f(x)$. It is the **union** of the indicated subdomains.

f) Give the **range** of f using interval form (the form with parentheses and/or brackets). (2 points)

$\text{Range}(f) = \boxed{[1,4]}$, which gives the ***y*-coordinates** "picked up" by the graph.

12) Let $s(t) = 3t^3 - t^2$. Find the average rate of change of s from $t = -1$ to $t = 2$. Assume that t is time measured in seconds and $s(t)$ is the position of a particle measured in meters. (The particle is moving along a coordinate line.) Write the appropriate unit in your final answer.

<u>Note</u>: You are finding the average velocity of the particle between $t = -1$ seconds and $t = 2$ seconds; we are allowing negative values for t. (7 points)

s is a polynomial function, so the usual formula $\dfrac{s(b)-s(a)}{b-a}$ applies.

$$\frac{s(2)-s(-1)}{2-(-1)} = \frac{\left[3(2)^3-(2)^2\right]-\left[3(-1)^3-(-1)^2\right]}{3} = \frac{20-(-4)}{3} = \frac{24}{3} = \boxed{8\ \frac{\text{m}}{\text{sec}}}$$

13) Let $f(x) = \sqrt{x}$. Simplify the difference quotient completely:

$$\frac{f(x+h)-f(x)}{h}$$

Hint: You will need to rationalize a numerator. (9 points)

• When you see $f(x+h)$, think "**substitution**."

$$\frac{f(x+h)-f(x)}{h} = \frac{\sqrt{x+h}-\sqrt{x}}{h} \quad \text{(Rationalize the numerator.)}$$

$$= \frac{\left(\sqrt{x+h}-\sqrt{x}\right)}{h}\cdot\frac{\left(\sqrt{x+h}+\sqrt{x}\right)}{\left(\sqrt{x+h}+\sqrt{x}\right)} = \frac{\left(\sqrt{x+h}\right)^2-\left(\sqrt{x}\right)^2}{h\left(\sqrt{x+h}+\sqrt{x}\right)} = \frac{(x+h)-(x)}{h\left(\sqrt{x+h}+\sqrt{x}\right)}$$

$$= \frac{\cancel{x}+h\cancel{-x}}{h\left(\sqrt{x+h}+\sqrt{x}\right)} = \frac{\overset{(1)}{\cancel{h}}}{\underset{(1)}{\cancel{h}}\left(\sqrt{x+h}+\sqrt{x}\right)} = \boxed{\frac{1}{\sqrt{x+h}+\sqrt{x}}\ (h\neq 0)}$$

• As h approaches 0, this approaches $\dfrac{1}{2\sqrt{x}}$, which is $f'(x)$, the **derivative** of $f(x)$.