MIDTERM 2 SOLUTIONS

(CHAPTERS 2 AND 3: POLYNOMIAL, RATIONAL, EXP'L, LOG FUNCTIONS)
MATH 141 – SPRING 2025 – KUNIYUKI
150 POINTS TOTAL: 44 FOR PART 1, AND 106 FOR PART 2

PART 1: USING SCIENTIFIC CALCULATORS (44 PTS.)

1) The profit P (in dollars) for the Superdoom computer game company is given by P or $P(x) = -20x^2 + 400x - 1500$, where x is the number of game DVDs produced and sold. You may assume that the domain of P is $[0, \infty)$.
For parts a) and b), write units! (8 points total)

   a) Write and use a formula we used in class to find the number of DVDs (produced and sold) for which profit is maximized. (4 points)

   The graph of P versus x is a piece of a parabola opening downward. We want the ***x***-coordinate of the **vertex** (the maximum point) of the parabola.

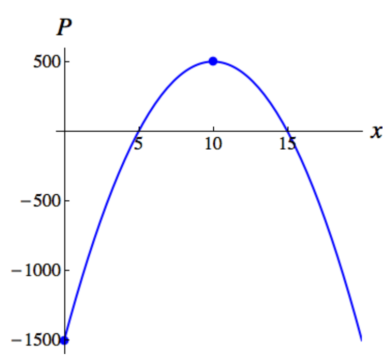
$$x = -\frac{b}{2a} = -\frac{(400)}{2(-20)} = \boxed{10 \text{ DVDs}}$$

   b) What is the corresponding maximum profit? (4 points)

   The **profit** when 10 DVDs are produced and sold is given by:

$$P(10) = -20(10)^2 + 400(10) - 1500 = -2000 + 4000 - 1500 = \boxed{\$500}$$

Note: Here is a graph of P versus x. We assume that the domain of P is $[0, \infty)$, though that may be technically unrealistic.



2) Consider $s(r) = r^3 - 3r^2 - 4r + 42$ in parts a) and b) below.
Hint: One of the zeros is -3. (16 points total)

a) Write the two other complex zeros of s in simplest, standard form. Show all work, as in class. Box in your answers! (13 points)

By the **Factor Theorem**, because -3 is a zero, $(r+3)$ is a factor of $s(r)$.
Use **Synthetic Division** to check this and to help us factor $s(r)$.

$$\begin{array}{r|rrrr} -3 & 1 & -3 & -4 & 42 \\ & & -3 & 18 & -42 \\ \hline & 1 & -6 & 14 & 0 \end{array}$$

Therefore, $s(r) = (r+3)(r^2 - 6r + 14)$.

Find the **zeros (roots)** of the quadratic factor, $r^2 - 6r + 14$.

- Observe: $a = 1,\ b = -6,\ c = 14.$
- The **discriminant** is: $b^2 - 4ac = (-6)^2 - 4(1)(14) = 36 - 56 = \mathbf{-20}.$
- $\mathbf{-20}$ is **not** a perfect square, and GCF = 1, so we **cannot** factor $r^2 - 6r + 14$ over $\mathbb{Z}$.
- Use the **Quadratic Formula (QF)**:

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-6) \pm \sqrt{\mathbf{-20}}}{2(1)} \quad \leftarrow \text{We know the discriminant.}$$

$$= \frac{6 \pm i\sqrt{20}}{2} = \frac{6 \pm i\sqrt{4 \cdot 5}}{2} = \frac{6 \pm 2i\sqrt{5}}{2} = \frac{6}{2} \pm \frac{2i\sqrt{5}}{2} = \boxed{3 \pm i\sqrt{5}}$$

b) Write the polynomial $s(r)$ as a product of three linear factors over $\mathbb{C}$, the set of complex numbers. We basically want the Linear Factorization Theorem (LFT) Form of the factorization. (3 points)

$$\boxed{s(r) = (r+3)\left[r - \left(3 + i\sqrt{5}\right)\right]\left[r - \left(3 - i\sqrt{5}\right)\right], \text{ or } (r+3)\left[r - 3 - i\sqrt{5}\right]\left[r - 3 + i\sqrt{5}\right]}$$

3) On the day of a child's birth, a deposit of \$5000 is made in a trust fund that pays 7.5% annual interest compounded continuously. Assuming there are no further deposits or withdrawals, how old will the child be when there is \$8000 in the account? Give **both** an **exact** answer (which may look ugly; you don't have to simplify it) and an **approximate** answer rounded off to three significant digits. Write units! (10 points)

Model: $f(t) = Pe^{rt}$

Solve $8000 = 5000e^{0.075t}$ for t

$$\frac{8000}{5000} = e^{0.075t}$$
$$1.6 = e^{0.075t}$$
$$\ln(1.6) = \ln\left(e^{0.075t}\right)$$
$$\ln(1.6) = 0.075t$$
$$t = \boxed{\begin{array}{l} \dfrac{\ln(1.6)}{0.075} \text{ years, or } \dfrac{1000\ln(1.6)}{75} \text{ years, or } \dfrac{40\ln\left(\frac{8}{5}\right)}{3} \text{ years, or} \\ \dfrac{40\left[\ln(8)-\ln(5)\right]}{3} \text{ years } (\textbf{exactly}) \\ \approx 6.27 \text{ years } (\textbf{approximately}) \end{array}}$$

4) Approximate $\log_8(179)$ to four decimal places. Show work by using a change-of-base formula we have discussed in class. (4 points)

Use the **Change of Base Formula**: $\log_8(179) = \dfrac{\ln(179)}{\ln(8)} \left[\text{or } \dfrac{\log(179)}{\log(8)}\right] \approx \boxed{2.4946}$.

5) Simplify $\dfrac{1}{2-7i}$ by writing the quotient in standard form. (6 points)

We will rationalize (or "real"-ize) the denominator by multiplying the numerator and the denominator by $3-5i$, the **complex conjugate** of the denominator.

$$\frac{1}{2-7i} = \frac{1}{(2-7i)} \cdot \frac{(2+7i)}{(2+7i)}$$

$\leftarrow$ In short: $(2)^2 + (7)^2 = 53$; the long way follows.

$$= \frac{2+7i}{4-49i^2} = \frac{2+7i}{4-49(-1)} = \frac{2+7i}{4+49} = \frac{2+7i}{53} = \frac{2}{53} + \frac{7i}{53} = \boxed{\frac{2}{53} + \frac{7}{53}i}$$

PART 2: NO CALCULATORS ALLOWED! (106 PTS.)

6) What is the vertex of the parabola given by $y=5(x+7)^2-4$? (2 points)

The vertex of the parabola given by $y=a(x-h)^2+k$ is (h,k). Here, it is $\boxed{(-7,-4)}$.

7) Fill in each blank below with ∞ or $-\infty$. (4 points total; 2 points each)

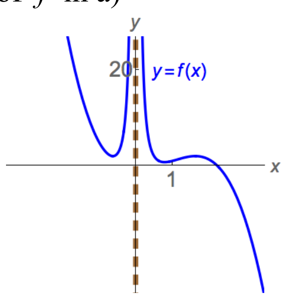
a) If $f(x)=-2x^3+5x^2-3+\dfrac{1}{x^2}$, then $\lim\limits_{x\to\infty} f(x)=$ $\underline{\boxed{-\infty}}$

$\lim\limits_{x\to\infty}\dfrac{1}{x^2}=0$, so focus on the polynomial part. The leading term is $-2x^3$, so the graph of $y=f(x)$ is a "**falling snake** in the long run." The graph "shoots down" without bound to the right.

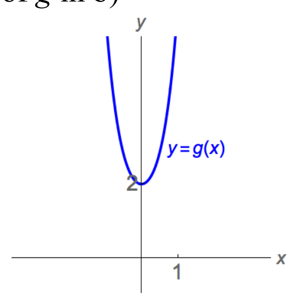
b) If $g(x)=2x^4+3x^2+2$, then $\lim\limits_{x\to-\infty} g(x)=$ $\underline{\boxed{\infty}}$

The leading term is $2x^4$, so the graph of $y=g(x)$ is an "**upward**-opening **bowl** in the long run." The graph "shoots up" without bound to the left.

Graph of f in a)



Graph of g in b)



8) How many turning points (TPs) can the graph of $y=f(x)$ have if f is a 4^{th}-degree polynomial function? (3 points)

Answer: $\boxed{\text{3 or 1}}$. Start with (degree – 1) and count down by twos; stop before the negative numbers. The graph must be a "**bowl** in the long run," so the number of TPs must be **odd**.

9) Simplify i^{447}. (2 points)

When we divide 447 by 4, we get a **remainder** of 3, so $i^{447} = i^3 = \boxed{-i}$.

(Observe that 444 is a nice multiple of 4; 447 is 3 more than that.)

10) Write the list of the possible rational zeros of f, where $f(x) = 7x^5 - 5x^3 + 9x + 3$, based on the Rational Zero Test (Rational Roots Theorem). You do <u>not</u> have to determine which of these candidates are, in fact, zeros. (6 points)

$f(x)$ is a polynomial with integer coefficients and a nonzero constant term, so the theorem applies.

$p: \pm 1, \pm 3$ $(\leftarrow$ Factors of 3, the constant term$)$

$q: \pm 1, \pm 7$ $(\leftarrow$ Factors of 7, the leading coefficient$)$

Think: $\pm \dfrac{1,\ 3}{1,\ 7}$ $(\leftarrow$ Informal$)$

$\dfrac{p}{q}: \boxed{\pm 1,\ \pm \dfrac{1}{7},\ \pm 3,\ \pm \dfrac{3}{7}}$ $(\leftarrow$ The candidates$)$

11) Use Long Division to perform the division: $\dfrac{8x^4 + 12x^3 - 6x^2 - 7x + 1}{4x^2 - 3}$.

Write your answer in the form: $(\text{polynomial}) + (\text{proper rational expression})$. (11 points)

The terms in red are deleted after their opposites are written in the subtraction process.

$$
\begin{array}{r|l}
 & 2x^2 + 3x \\
\cline{2-2}
4x^2 + 0x - 3 & 8x^4 + 12x^3 - 6x^2 - 7x + 1 \\
 & \cancel{8x^4 + 0x^3 - 6x^2} \\
 & -8x^4 - 0x^3 + 6x^2 \\
\cline{2-2}
 & \qquad 12x^3 + 0x^2 - 7x + 1 \\
 & \qquad \cancel{12x^3 + 0x^2 - 9x} \\
 & \qquad -12x^3 - 0x^2 + 9x \\
\cline{2-2}
 & \qquad\qquad\qquad 2x + 1
\end{array}
$$

$2x + 1$ has **degree** 1, which is less than 2, the **degree** of $4x^2 - 3$. Therefore, $2x + 1$ is an appropriate **remainder**.

Answer: $\boxed{2x^2 + 3x + \dfrac{2x + 1}{4x^2 - 3}}$

12) Find a fifth-degree polynomial (with real coefficients) written in descending powers of x that has the following properties: It has $3i$ and 0 as roots (or "zeros"), and 0 is a root ("zero") of multiplicity 3.

Hint: If a polynomial with real coefficients has $3i$ as a root ("zero"), what other complex number must also be a root ("zero")? (8 points)

$-3i$ must also be a root ("zero"). Use the **Factor Theorem**:

$$x^3(x-3i)(x+3i) = x^3(x^2+9) = \boxed{x^5+9x^3}$$

Any nonzero constant multiple, such as: $2x^3(x^2+9) = 2x^5+18x^3$, will work.

13) Consider $f(x) = 4x^7 + x^3 - 2x^2 + 9x - 3$. Using only Descartes's Rule of Signs, … (8 points total)

a) List the possible numbers of **positive** real zeros of f (accounting for multiplicity: double roots are counted twice, for example).

$$f(x) = +4x^7 + x^3 - 2x^2 + 9x - 3$$

has 3 variations in sign.

Counting down by twos, there are either $\boxed{3 \text{ or } 1}$ positive real zeros.
Note: In fact, there is only one (with multiplicity 1): about 0.356182.

b) List the possible numbers of **negative** real zeros of f (accounting for multiplicity: double roots are counted twice, for example).

$$\begin{aligned} f(-x) &= 4(-x)^7 + (-x)^3 - 2(-x)^2 + 9(-x) - 3 \\ &= -4x^7 - x^3 - 2x^2 - 9x - 3 \end{aligned}$$

has 0 variations in sign.

There must be $\boxed{0}$ negative real zeros.

14) Let $f(x) = \dfrac{x^2 - x - 6}{2x^2 + 5x + 2}$. Consider the graph of $y = f(x)$. If an answer to a part below is none, write "NONE." Box in the answers! (20 points total)

$f(x)$ is rational with (nonzero) polynomial numerator and denominator.

a) Factor the numerator and the denominator of $\dfrac{x^2 - x - 6}{2x^2 + 5x + 2}$, and simplify the expression. (5 points)

$$f(x) = \frac{x^2 - x - 6}{2x^2 + 5x + 2} = \boxed{\frac{(x+2)(x-3)}{(2x+1)(x+2)}} = \frac{\overset{(1)}{\cancel{(x+2)}}(x-3)}{(2x+1)\underset{(1)}{\cancel{(x+2)}}},\ (x+2 \neq 0) = \boxed{\frac{x-3}{2x+1},\ (x \neq -2)}$$

b) Find the x-intercept(s), if any. (3 points)

From a): $f(x)=\frac{x-3}{2x+1}$, $(x\neq -2)$. Set y or $f(x)=0$, and solve for x:

$$0=\frac{x-3}{2x+1},\ (x\neq -2)$$

$$x-3=0, \qquad \left(x\neq -2,-\frac{1}{2}\right)$$

$$x=3$$

The x-intercept is at $\boxed{(3,0)}$.

c) Find the y-intercept, if any. (3 points)

$0\in \mathrm{Dom}(f)$, so set $x=0$ in the simplified expression from a):

$$f(x) = \frac{x-3}{2x+1},\ (x\neq -2) \Rightarrow f(0) = \frac{(0)-3}{2(0)+1} = -3$$

The y-intercept is at $\boxed{(0,-3)}$.

d) Give the x-coordinate(s) of the hole(s), if any.
(Holes correspond to "removable discontinuities.") (3 points)

See a). The original numerator and denominator have $(x+2)$ as a **common variable factor**, and **all** $(x+2)$ factors are **entirely canceled (divided) out in the denominator**. There is a **hole** at $x=\boxed{-2}$.

<u>Note</u>: The hole is at the point $\left(-2,\frac{5}{3}\right)$, because:

$$\lim_{x\to -2} f(x) = \lim_{x\to -2}\frac{x-3}{2x+1} = \frac{(-2)-3}{2(-2)+1} = \frac{5}{3}$$

e) Find the equation(s) of the vertical asymptote(s) (VAs), if any. (3 pts.)

Find the **real zeros of the denominator** in the <u>simplified</u> expression in a):

$$2x+1=0$$

$$x=-\frac{1}{2}$$

The $(2x+1)$ factors are **not** entirely canceled (divided) out in the denominator after simplifying in a). The one **VA** has equation: $\boxed{x=-\frac{1}{2}}$.

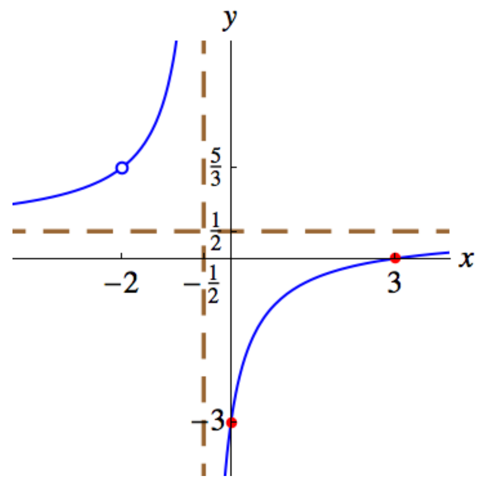
f) Find the equation of the horizontal asymptote (HA), if any. (3 points)

The degrees of the numerator and denominator of $\frac{x^2-x-6}{2x^2+5x+2}$ are equal (both are 2), so there is one **HA**: $y =$ the ratio of their leading coefficients.

The HA has equation: $\boxed{y=\frac{1}{2}}$.

<u>Note</u>: We could have used the simplified version: $f(x)=\frac{x-3}{2x+1},\ x\neq -2$.
The degrees of the numerator and denominator are equal (both are 1).

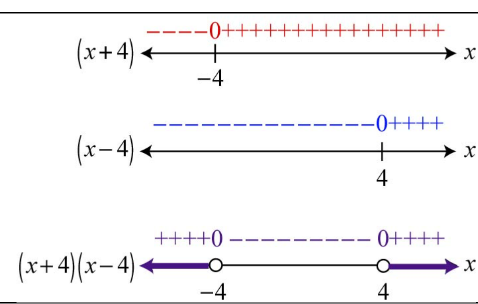
Here is the graph of $y=f(x)$:



15) Write the domain of f, where $f(x)=\frac{1}{\sqrt{x^2-16}}$, using interval form (the form using parentheses and/or brackets). (5 points)

- Due to the **even-root** radical (the index, 2, is even), the radicand must be **nonnegative**: $x^2-16\geq 0$.
- We further require $x^2-16\neq 0$ so as to **avoid a zero denominator**.
- The domain is the solution set of the inequality: $x^2-16>0$.
- See the Notes for Section 2.7 to see various methods of solution.

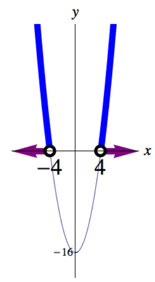
<u>Method 1</u>: Sign Chart Method

$x^2-16>0$ $(x+4)(x-4)>0$	

Answer: $\boxed{(-\infty,-4)\cup(4,\infty)}$

Method 2: Parabola Method (for Quadratic Inequalities)

The graph of $y = x^2 - 16$ is an **upward**-opening **parabola** with ***x*-intercepts** at $(-4, 0)$ and $(4, 0)$. To solve $x^2 - 16 > 0$, we "pick up" the ***x*-coordinates** corresponding to the parts of the parabola lying strictly **above the *x*-axis**.



Answer: $\boxed{(-\infty, -4) \cup (4, \infty)}$

Methods 3, 4: Test Value / Interval / Window Method; Evaluation or Factoring

		−4		**4**	
Test x-values	-5		0		5
Value of $x^2 - 16$ Sign Analysis: $(x+4)(x-4)$	$f(-5) = 9$ $(-)(-)$	0	$f(0) = -16$ $(+)(-)$	0	$f(5) = 9$ $(+)(+)$
Sign of $x^2 - 16$	$+$	0	$-$	0	$+$

Answer: $\boxed{(-\infty, -4) \cup (4, \infty)}$

16) Write the domain of f, where $f(x) = e^x$, using interval form (the form using parentheses and/or brackets). (1 point)

$\text{Dom}(f) = \boxed{(-\infty, \infty)}$.

17) Write the domain of g, where $g(x) = \ln(x)$, using interval form (the form using parentheses and/or brackets). (1 point)

$\text{Dom}(g) = \boxed{(0, \infty)}$.

18) Simplify the following: (4 points total; 2 points each)

a) $\log\left(\frac{1}{100}\right) = \log_{10}\left(10^{-2}\right) = \boxed{-2}$ [by Inverse Properties]; $10^{\boxed{-2}} = \frac{1}{10^2} = \frac{1}{100}$

b) $\log_8(2) = \log_8\left(\sqrt[3]{8}\right) = \log_8\left(8^{1/3}\right) = \boxed{\frac{1}{3}}$ [by Inverse Properties]; $8^{\boxed{1/3}} = \sqrt[3]{8} = 2$

19) Which of the following is equivalent to $\left[\log(x)\right]^5 + 2^{x+3}$? Box in one: (3 points)

a) $\left[\log(x)\right]^5 + 6\left(2^x\right)$

$\boxed{\text{b) } \left[\log(x)\right]^5 + 8\left(2^x\right)}$ $\left[\text{Note: } 2^{x+3} = 2^x \cdot 2^3 = 2^x \cdot 8 = 8\left(2^x\right).\right]$

c) $5\log(x) + 2^{x+3}$ [Note: We cannot "smack down" the 5.]

20) What must be true of $\log_2(20)$? Box in one: (3 points)

a) It is between 2 and 3.

b) It is between 3 and 4.

$\boxed{\text{c) It is between 4 and 5.}}$

- $\log_2(x)$ increases on its domain, $(0, \infty)$.
- $\log_2(16) = 4$, since $2^4 = 16$.
- $\log_2(32) = 5$, since $2^5 = 32$.
- Therefore, $4 < \log_2(20) < 5$.

21) Expand and evaluate where appropriate: $\ln\left(\dfrac{x^2 y^3}{e^4 \cdot \sqrt[5]{z}}\right)$. Assume $x, y, z > 0$.

(10 points)

$$\ln\left(\frac{x^2 y^3}{e^4 \cdot \sqrt[5]{z}}\right) = \ln\left(x^2 y^3\right) - \ln\left(e^4 \cdot \sqrt[5]{z}\right)$$

[by the Quotient Rule for Logarithms]

$$= \left[\ln\left(x^2\right) + \ln\left(y^3\right)\right] - \left[\ln\left(e^4\right) + \ln\left(\sqrt[5]{z}\right)\right]$$

[by the Product Rule for Logarithms; watch grouping symbols!]

$$= \ln\left(x^2\right) + \ln\left(y^3\right) - \ln\left(e^4\right) - \ln\left(\sqrt[5]{z}\right)$$

[Note the last "−."]

$$= \ln\left(x^2\right) + \ln\left(y^3\right) - 4 - \ln\left(z^{1/5}\right)$$

$\left[\text{Note: } \ln\left(e^4\right) = 4, \text{ by the Inverse Properties}\right]$

$$= \boxed{2\ln(x) + 3\ln(y) - 4 - \frac{1}{5}\ln(z)}$$

[by the Power / Smackdown Rule for Logarithms]

Note: If we allow $x < 0$, then we would need $2\ln|x|$ instead of $2\ln(x)$.

22) Find all real solution(s) of the equation: $\log(3x+1)-\log(x-1)=\log(x+2)$.
Write the solution set. Show all work, as in class; do not use trial-and-error!
(10 points)

$$\log(3x+1)-\log(x-1)=\log(x+2)$$

$$\log\left(\frac{3x+1}{x-1}\right)=\log(x+2) \qquad \text{[by the Quotient Rule for Logs]}$$

$$\frac{3x+1}{x-1}=x+2 \qquad \text{[by the One-to-One Properties; see (*)]}$$

$$3x+1=(x+2)(x-1),\quad x\neq 1 \qquad \text{[1 will fail the check, anyway.]}$$

$$3x+1=x^2+x-2$$

$$0=x^2-2x-3 \qquad \text{[Isolate 0 on one side.]}$$

$$0=(x+1)(x-3)$$

$$x+1=0 \quad \text{or} \quad x-3=0$$

$$\cancel{x=-1} \quad \text{or} \quad x=3$$

Checking tentative solutions is required when solving log equations. Only $x=3$ checks out in the original equation; we only take logs of positive values.

Check $x=-1$:	Check $x=3$:
$\log[3(-1)+1]-\log(-1-1)=\log(-1+2)$ $\cancel{\log(-2)}-\cancel{\log(-2)}=\log(1)$ not real not real (Check fails.)	$\log[3(3)+1]-\log(3-1)=\log(3+2)$ $\log(10)-\log(2)=\log(5)$ $\log\left(\frac{10}{2}\right)=\log(5)$ [by the Quotient Rule for Logs] $\log(5)=\log(5)$ (Checks out.)

Solution set $=\boxed{\{3\}}$.

Note 1 (*): We could have performed base-10 exponentiation on both sides:

$$10^{\log_{10}\left(\frac{3x+1}{x-1}\right)}=10^{\log_{10}(x+2)}$$

$$\frac{3x+1}{x-1}=x+2$$

Note 2: Domain issues are considered at the end, when we checked our tentative solutions.

23) Find the real solution of the equation: $\frac{1}{9}\left(3^{4x+1}\right)=3$. The solution is a rational number, and you must write it in simplified form. Show all work, as in class; do <u>not</u> use trial-and-error! (5 points)

<u>Method 1</u>

$$\frac{1}{9}\left(3^{4x+1}\right)=3$$

$$3^{4x+1}=27 \qquad \left[\text{Isolate the exponential.}\right]$$

$$3^{4x+1}=3^3$$

$$4x+1=3 \qquad \left[\text{Use the One-to-One Property of the } 3^x \text{ function.}\right]$$

$$4x=2$$

$$x=\frac{2}{4}$$

$$x=\frac{1}{2} \quad \Rightarrow \quad \text{Solution set } = \boxed{\left\{\frac{1}{2}\right\}}.$$

<u>Method 2</u>

$$\frac{1}{9}\left(3^{4x+1}\right)=3$$

$$3^{4x+1}=27 \qquad \left[\text{Isolate the exponential.}\right]$$

$$\log_3\left(3^{4x+1}\right)=\log_3(27) \qquad \left[\text{Note: } \log_3(27)=3, \text{ since } 3^3=27.\right]$$

$$4x+1=3 \qquad \left[\text{Use the Inverse Properties.}\right]$$

$$4x=2$$

$$x=\frac{2}{4}$$

$$x=\frac{1}{2} \quad \Rightarrow \quad \text{Solution set } = \boxed{\left\{\frac{1}{2}\right\}}.$$