MIDTERM 2 SOLUTIONS

(CHAPTERS 2 AND 3: POLYNOMIAL, RATIONAL, EXP'L, LOG FUNCTIONS)
MATH 141 – SPRING 2026 – KUNIYUKI
150 POINTS TOTAL: 56 FOR PART 1, AND 94 FOR PART 2

PART 1: USING SCIENTIFIC CALCULATORS (56 PTS.)

1) Write the "Vertex Form" of the equation of the parabola in the usual *xy*-plane that opens downward, that has $(-1, 4)$ as its vertex, and that passes through the point $(1, -16)$. (7 points)

Vertex Form is given by: $y = a(x-h)^2 + k$. **Don't forget the square!**

Here, the **vertex** $(h, k) = (-1, 4)$. We now have:

$$y = a\left(x - (-1)\right)^2 + 4$$
$$y = a(x+1)^2 + 4$$

Now apply the **Basic Principle of Graphing** and use the fact that $(x = 1,\ y = -16)$ must satisfy the equation of the parabola:

$$y = a(x+1)^2 + 4 \quad \Rightarrow$$
$$(-16) = a\left((1)+1\right)^2 + 4$$
$$-16 = a(2)^2 + 4$$
$$-16 = 4a + 4$$
$$-20 = 4a$$
$$a = -5 \quad \Rightarrow \quad \boxed{y = -5(x+1)^2 + 4}$$

2) An astronaut kicks a ball over a flat region of a (very) distant moon. The height of the ball in feet is given by: $h(t) = -3t^2 + 18t + 2 \; (\text{if } t \geq 0)$, where t is the amount of time in seconds since the ball was kicked. (The formula is relevant up until the moment the ball hits the ground.) Write units in your answers! (19 points total)

a) Write and use a formula we used in class to find how much time it takes (since the ball was kicked) for the ball to reach its maximum height. (4 points)

The graph of $h(t)$ versus t is a parabolic piece opening downward. We want the t-coordinate of the **vertex** (the maximum point) of the parabola.

$$t = -\frac{b}{2a} = -\frac{(18)}{2(-3)} = \boxed{3 \text{ seconds}} \; (\text{after the ball was kicked})$$

b) What is the maximum height achieved by the ball? (4 points)

The height of the ball 3 seconds after it was kicked is given by:

$$h(3) = -3(3)^2 + 18(3) + 2 = -27 + 54 + 2 = \boxed{29 \text{ feet}}$$

c) What was the height of the ball at the time it was kicked? (3 points)

$$h(0) = \boxed{2 \text{ feet}}$$

d) How much time does it take (since the ball was kicked) for the ball to hit the ground? Give an exact answer and also round it off to three significant digits. (8 points)

Solve $h(t) = 0$ for $t > 0$; that is, $-3t^2 + 18t + 2 = 0$ for $t > 0$.

We want the **positive** real zero of $-3t^2 + 18t + 2$. $(a = -3, \; b = 18, \; c = 2)$.

Its **discriminant** is: $b^2 - 4ac = (18)^2 - 4(-3)(2) = 324 + 24 = \mathbf{348}$.

348 is **not** a perfect square, and GCF = 1, so we **cannot** factor $h(t)$ over $\mathbb{Z}$ using the "Guess Method." Let's use the **Quadratic Formula (QF)**.

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(18) \pm \sqrt{\mathbf{348}}}{2(-3)}$$ ← We found the **discriminant** already.

$$= \frac{-18 \pm 2\sqrt{87}}{-6} = \frac{-18}{-6} \pm \frac{2\sqrt{87}}{-6} = 3 \pm \left(-\frac{\sqrt{87}}{3}\right) = 3 \mp \frac{\sqrt{87}}{3} \text{ or } \frac{9 \mp \sqrt{87}}{3}$$

Observe: $3 - \frac{\sqrt{87}}{3}$ or $\frac{9 - \sqrt{87}}{3} \approx -0.109 < 0$, so we **reject** this zero.

The ball hits the ground in $\boxed{\left(3 + \frac{\sqrt{87}}{3}\right) \text{ or } \frac{9 + \sqrt{87}}{3} \text{ seconds} \approx 6.11 \text{ seconds}}$.

3) Consider $f(t) = t^3 - 7t^2 + 17t - 14$ in parts a) and b) below.
Hint: One of the zeros is 2. (16 points total)

a) Write the two other complex zeros of f in simplest, standard form. Show all work, as in class. Box in your answers! (13 points)

By the **Factor Theorem**, because 2 is a zero, $(t-2)$ is a factor of $f(t)$.

Use **Synthetic Division** to check this and to help us factor $f(t)$.

2	1	−7	17	−14
		2	−10	14
	1	−5	7	0

Therefore, $f(t) = (t-2)(t^2 - 5t + 7)$.

Find the zeros (roots) of the quadratic factor, $t^2 - 5t + 7$.

- We have: $a = 1,\ b = -5,\ c = 7$.
- The **discriminant** is: $b^2 - 4ac = (-5)^2 - 4(1)(7) = 25 - 28 = \mathbf{-3}$.
- $\mathbf{-3}$ is **not** a perfect square, and GCF = 1. We **cannot** factor $t^2 - 5t + 7$ over $\mathbb{Z}$ using the "Guess Method."
- We can find the zeros (roots) by using the **Quadratic Formula (QF)**:

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-5) \pm \sqrt{-3}}{2(1)} = \frac{5 \pm i\sqrt{3}}{2} = \boxed{\frac{5}{2} \pm \frac{\sqrt{3}}{2} i}$$

- We already verified that 2 was a zero.

b) Write the polynomial $f(t)$ as a product of three linear factors over $\mathbb{C}$, the set of complex numbers. We basically want the Linear Factorization Theorem (LFT) Form of the factorization. (3 points)

$$\boxed{\begin{gathered} f(t) = (t-2)\left[t - \left(\frac{5}{2} + \frac{\sqrt{3}}{2} i\right)\right]\left[t - \left(\frac{5}{2} - \frac{\sqrt{3}}{2} i\right)\right], \text{ or} \\ (t-2)\left[t - \frac{5}{2} - \frac{\sqrt{3}}{2} i\right]\left[t - \frac{5}{2} + \frac{\sqrt{3}}{2} i\right] \end{gathered}}$$

4) On the day of a child's birth, a deposit of $2000 is made in a trust fund that pays 5.5% annual interest compounded continuously. Assuming there are no further deposits or withdrawals, how old will the child be when there is $15,000 in the account? Give **both** an **exact** answer (which may look ugly; you don't have to simplify it) and an **approximate** answer rounded off to three significant digits. Write units! (10 points)

Model: $f(t) = Pe^{rt}$

Solve $15{,}000 = 2000e^{0.055t}$ for t

$$\frac{15{,}000}{2000} = e^{0.055t}$$

$$7.5 = e^{0.055t} \quad [\leftarrow 7.5 \text{ is exact.}]$$

$$\ln(7.5) = \ln\left(e^{0.055t}\right)$$

$$\ln(7.5) = 0.055t$$

$$t = \boxed{\frac{\ln(7.5)}{0.055} \text{ or } \frac{\ln\left(\frac{15}{2}\right)}{\frac{11}{200}} \text{ or } \frac{200\left[\ln(15) - \ln(2)\right]}{11} \text{ years} \quad (\text{exactly})}$$

$$\approx \boxed{36.6 \text{ years} \quad (\text{approximately})}$$

5) Approximate $\log_5(1234)$ to four decimal places. Show work by using a change-of-base formula we have discussed in class. (4 points)

Use the **Change of Base Formula**: $\log_5(1234) = \frac{\ln(1234)}{\ln(5)} \left[\text{or } \frac{\log(1234)}{\log(5)}\right] \approx \boxed{4.4227}$

PART 2: NO CALCULATORS ALLOWED! (94 POINTS)

6) Use Long Division to perform the division: $\dfrac{6x^5+4x^4-15x^2-14x}{2x^3-5}$.

Write your answer in the form: $(\text{polynomial})+(\text{proper rational expression})$. (11 points)

The terms in red are deleted after their opposites are written in the subtraction process.

$$
\begin{array}{r|l}
 & \qquad\qquad\qquad\quad 3x^2 + 2x \\
\hline
2x^3+0x^2+0x-5 & 6x^5+4x^4+0x^3-15x^2-14x \\
 & \sout{6x^5+0x^4+0x^3-15x^2} \\
 & \underline{-6x^5-0x^4-0x^3+15x^2} \\
 & \qquad\quad 4x^4+0x^3+\ 0x^2-14x \\
 & \qquad\quad \sout{4x^4+0x^3+\ 0x^2-10x} \\
 & \qquad\quad \underline{-4x^4-0x^3-\ 0x^2+10x} \\
 & \qquad\qquad\qquad\qquad\qquad -\ 4x
\end{array}
$$

$-4x$ has **degree** 1, which is less than 3, the **degree** of $2x^3-5$. Therefore, $-4x$ is an appropriate **remainder**.

Answer: $\boxed{3x^2+2x+\dfrac{-4x}{2x^3-5},\ \text{or}\ 3x^2+2x-\dfrac{4x}{2x^3-5}}$.

7) Match the equations with their corresponding graphs by writing the appropriate letters in the blanks. Assume that there are no other turning (turnaround) points outside the "scope" of the figures below. The x- and y-axes are not necessarily scaled the same way within and between graphs. (4 points)

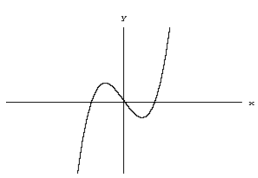
The graph of $y = x^4 + 3x^2 - x + 1$ is Graph __**D**__.
("Upward-opening bowl" in the "long run")

The graph of $y = -x^5 + 2x^2 + 1$ is Graph __**B**__.
("Falling snake")

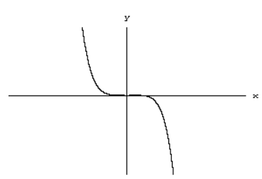
The graph of $y = x^5 - 10x^3 + 9x + 1$ is Graph __**C**__.
("Rising snake" with four turning (turnaround) points; this could not be the graph of the third-degree polynomial function.)

The graph of $y = x^3 - 8x + 1$ is Graph __**A**__.
("Rising snake" with only two turning (turnaround) points)

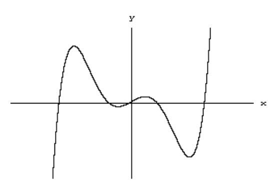
Graph A



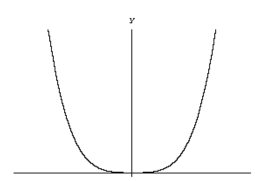
Graph B



Graph C



Graph D



8) If $f(x)$ is a nonzero polynomial with real coefficients such that one of its zeros is $3+5i$, what other complex number must also be a zero of $f(x)$? (1 point).

$\boxed{3-5i}$, the **complex conjugate** of $3+5i$, must also be a **zero**.

9) Fill in the blank:
If $f(x) = (x-4)^3$, then 4 is a zero of f with multiplicity _$\boxed{3}$_. (1 point)

10) Write the list of the possible rational zeros of f, where $f(x) = 7x^5 + 12x^3 - 4x^2 + 2$, based on the Rational Zero Test (Rational Roots Theorem). You do <u>not</u> have to determine which of these candidates are, in fact, zeros. (6 points)

$f(x)$ is a polynomial with integer coefficients and a nonzero constant term, so the theorem applies.

$p: \pm 1, \pm 2$ $(\leftarrow$ Factors of 2, the constant term$)$

$q: \pm 1, \pm 7$ $(\leftarrow$ Factors of 7, the leading coefficient$)$

Think: $\dfrac{\pm 1, \pm 2}{\pm 1, \pm 7}$ $(\leftarrow$ Informal$)$

$$\frac{p}{q}: \boxed{\pm 1, \pm \frac{1}{7}, \pm 2, \pm \frac{2}{7}}$$

(The candidates)

11) Simplify i^{447}. (2 points)

When we divide 447 by 4, we get a **remainder** of 3, so $i^{447} = i^3 = \boxed{-i}$.
(Observe that 444 is a nice multiple of 4; 447 is 3 more than that.)

12) Consider $f(x) = 7x^5 - 3x^4 - 2x + 5$. Using only Descartes's Rule of Signs, … (8 points total)

a) List the possible numbers of **positive** real zeros of f (accounting for multiplicity: double roots are counted twice, for example). (3 points)

$$f(x) = +7x^5 - 3x^4 - 2x + 5$$ has 2 **variations in sign**.

Counting down by twos, there are either $\boxed{2 \text{ or } 0}$ **positive** real zeros.
Note: In fact, there are no positive real zeros.

b) List the possible numbers of **negative** real zeros of f (accounting for multiplicity: double roots are counted twice, for example).
Show work, as in class. (5 points)

$$f(-x) = 7(-x)^5 - 3(-x)^4 - 2(-x) + 5$$
$$f(-x) = -7x^5 - 3x^4 + 2x + 5$$ has 1 **variation in sign**.

There must be $\boxed{1}$ **negative** real zero. (It is about -0.922371.)

13) Consider the graph of $y = \frac{(x+1)^2(x+2)}{(x+1)(x+2)^4}$ in the usual *xy*-plane.

If an answer to a part below is none, write "NONE." Box in the answers! (6 points total; 2 each)

Let $f(x) = \frac{(x+1)^2(x+2)}{(x+1)(x+2)^4} = \frac{x+1}{(x+2)^3}\ (x \neq -1)$.

a) Give the *x*-coordinate(s) of the hole(s), if any. (Holes correspond to "removable discontinuities.")

The numerator and denominator have $(x+1)$ as a **common variable factor**, and **all** $(x+1)$ factors are **entirely canceled (divided) out in the denominator** after simplifying. There is a **hole** at $\boxed{x=-1}$.

Note: The hole is at $(-1, 0)$, since $\lim_{x \to -1} f(x) = \lim_{x \to -1} \frac{x+1}{(x+2)^3} = \frac{-1+1}{(-1+2)^3} = 0$.

b) Find the equation(s) of the vertical asymptote(s) (VAs), if any.

Find the **real zeros of the denominator** of the **simplified** expression $\frac{x+1}{(x+2)^3}\ (x \neq -1)$; observe that the $(x+2)$ factors are **not** entirely **canceled (divided)** out in the **denominator**.

$$(x+2)^3 = 0 \Leftrightarrow x+2=0 \Leftrightarrow x=-2$$

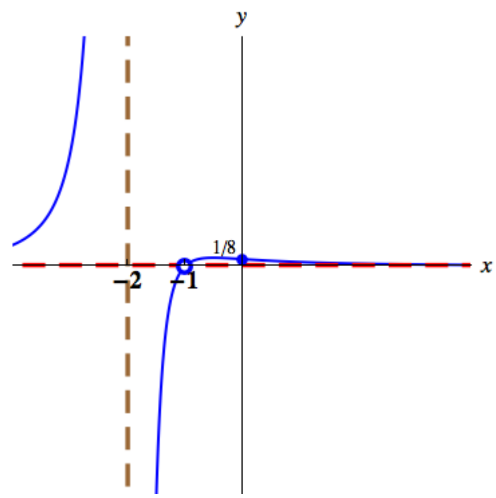
The one **VA** has equation: $\boxed{x=-2}$.

c) Find the equation of the horizontal asymptote (HA), if any.

In the simplified expression $\frac{x+1}{(x+2)^3}\ (x \neq -1)$, the **degree of the numerator** (1) is **less than** the **degree of the denominator** (3), so the expression is **"bottom-heavy" (proper)** in degree. The **HA** has equation: $\boxed{y=0}$.

Note 1: There are **no *x*-intercepts**, since $f(x)$ is **never 0**. In the simplified expression, the sole real zero of the numerator (-1) is **not** in the **domain** of f. The **hole** at $(-1, 0)$ does **not** count as an *x*-intercept.

Note 2: The ***y*-intercept** is at $\left(0, \frac{1}{8}\right)$, since $f(0) = \frac{1}{8}$.



14) Consider the graph of $y=\dfrac{3x^2+1}{6x^2-6}$ in the usual *xy*-plane. If an answer to a part below is none, write “NONE.” Box in the answers! (14 points total)

Let $f(x)=\dfrac{3x^2+1}{6x^2-6}$, which is rational with polynomial numerator and denominator.

a) Find the *x*-intercept(s), if any. (3 points)

Set y or $f(x)=0$ and solve for *x*. We want the **real zeros** of f.

$$0=\frac{3x^2+1}{6x^2-6}$$
$$0=3x^2+1 \quad \left(\text{provided } 6x^2-6\neq 0\right)$$
$$-1=3x^2 \quad \left(\text{provided } 6x^2-6\neq 0\right)$$
$$x^2=-\frac{1}{3} \quad \left(\text{provided } 6x^2-6\neq 0\right)$$

This has **no** real solutions, so there are **no** *x*-intercepts. Answer: $\boxed{\text{NONE}}$.

b) Find the *y*-intercept, if any. (3 points)

Set $x=0$. $f(0)=\dfrac{3(0)^2+1}{6(0)^2-6}=\dfrac{1}{-6}=-\dfrac{1}{6} \Rightarrow \boxed{\left(0,-\dfrac{1}{6}\right)}$ is the ***y*-intercept**.

c) Find the equation(s) of the vertical asymptote(s) (VAs), if any. (5 points)

Find the **real zeros of the denominator**, $6x^2-6$:

$$6x^2-6=0$$

Note: You may also divide both sides by 6.

$$6\left(x^2-1\right)=0$$
$$6(x+1)(x-1)=0$$
$$x=-1 \text{ or } x=1$$

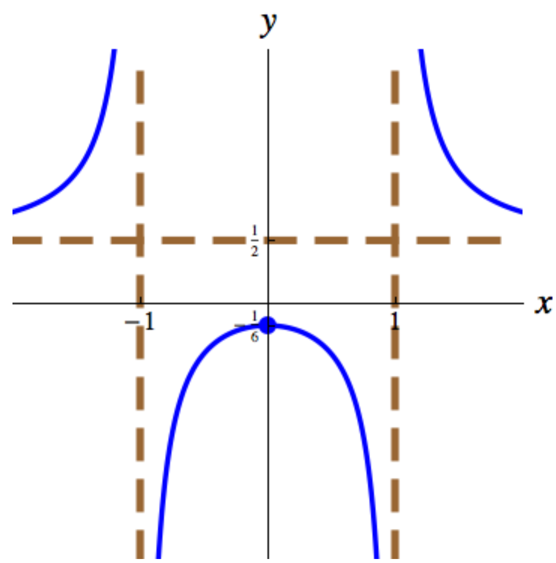
These are **not** zeros of the numerator, $3x^2+1$, so $(x+1)$ and $(x-1)$ are **not** factors of the numerator that could lead to holes. In fact, $\dfrac{3x^2+1}{6x^2-6}$ is **simplified**.

The **VAs** have equations: $\boxed{x=-1 \text{ and } x=1}$.

d) Find the equation of the horizontal asymptote (HA), if any. (3 points)

The **HA** is at $\boxed{y=\dfrac{1}{2}}$, the **ratio of the leading coefficients** of the numerator and denominator; note that $\dfrac{3}{6}=\dfrac{1}{2}$. This is because the **degree of the numerator** of $\dfrac{3x^2+1}{6x^2-6}$ is **equal** to the **degree of the denominator** (2).

Here is the graph of $y=f(x)$; observe that f is **even**:



15) Write the domain of f, where $f(x)=\sqrt[4]{x^2-16}$ using interval form (the form using parentheses and/or brackets). (5 points)

- We have an **even**-root radical (the index, 4, is **even**), so we require that the **radicand** be **nonnegative**: $f(x)$ is real $\Leftrightarrow$ $x^2-16\geq 0$. The **domain** is the **solution set of this inequality**.
- See the Notes for Section 2.7 on $f(x)=\sqrt{x^2-9}$.
- Key differences here: x^2-16 factors as $(x+4)(x-4)$, it has -4 and 4 as zeros, and $\text{Dom}(f)=\boxed{(-\infty,-4]\cup[4,\infty)}$. Consider the graph of $y=x^2-16$.

16) Write the **domain** of f, where $f(x)=10^x$, in interval form (the form using parentheses and/or brackets). (1 point). $\boxed{(-\infty,\infty)}$

17) Write the **range** of f from 16). (1 point). $\boxed{(0,\infty)}$

18) Write the **domain** of f, where $f(x)=\ln(x)$, in interval form (the form using parentheses and/or brackets). (1 point) . $\boxed{(0,\infty)}$

19) Write the **range** of f from 18). (1 point). $\boxed{(-\infty,\infty)}$

20) Simplify the following: (6 points total; 2 points each)

a) $\log_{16}(2)$

$\log_{16}(2)=\boxed{\frac{1}{4}}$. This is because $\sqrt[4]{16}=2$, and therefore $16^{\boxed{1/4}}=2$.

b) $\log_8(8^{12})$

$\log_8(8^{12})=\boxed{12}$ by the Inverse Properties for logarithms. Also, $8^{\boxed{12}}=8^{12}$.

c)

$$\log_3\left(\frac{1}{27}\right) = \log_3\left(\frac{1}{3^3}\right) = \log_3\left(3^{-3}\right) = \boxed{-3} \text{ (by the Inverse Properties)}$$

$$3^{\boxed{-3}} = \frac{1}{3^3} = \frac{1}{27}$$

21) Expand and evaluate where appropriate: $\ln\left[\frac{e^3\left(\sqrt{x}\right)}{y^2z^5}\right]$. $x, y, z > 0$. (10 points)

$$\ln\left[\frac{e^3\left(\sqrt{x}\right)}{y^2z^5}\right] = \ln(e^3) + \ln\left(\sqrt{x}\right) - \ln(y^2) - \ln(z^5)$$

[by the Product and Quotient Rules for Logarithms]

$$= 3 + \ln(x^{1/2}) - \ln(y^2) - \ln(z^5)$$

[Observe: $\ln(e^3)=3$, by the Inverse Properties]

$$= \boxed{3 + \frac{1}{2}\ln(x) - 2\ln(y) - 5\ln(z)}$$

[by the Power / "Smackdown" Rule for Logarithms]

Note: If $y<0$ were allowed, then write $-2\ln|y|$ instead of $-2\ln(y)$.

22) Find all real solution(s) (in simplified form) of the equation: $4^{x-1} = 16^{2x}$.
Write the solution set. Show all work, as in class; do not use trial-and-error!
(7 points)

$$4^{x-1} = 16^{2x}$$
$$4^{x-1} = \left(4^2\right)^{2x}$$
$$4^{x-1} = 4^{4x}$$
$$x - 1 = 4x \quad \text{[by the One-to-One Properties]}$$
$$-1 = 3x$$
$$x = -\frac{1}{3}$$

Solution set = $\boxed{\left\{-\frac{1}{3}\right\}}$.

23) Find all real solution(s) of the equation: $\log_5(x) - \log_5(x-100) = 1$.
Write the solution set. Show all work, as in class; do not use trial-and-error!
(9 points)

Note: We will consider domain issues at the end.

$$\log_5(x) - \log_5(x-100) = 1$$
$$\log_5\left(\frac{x}{x-100}\right) = 1 \quad \text{[by the Quotient Rule for Logs]}$$
$$5^1 = \frac{x}{x-100} \quad \text{["Zigzagged" to Exponential Form; see Note (*).]}$$
$$5 = \frac{x}{x-100}$$
$$5(x-100) = x$$
$$5x - 500 = x$$
$$4x = 500$$
$$x = 125$$

Checking tentative solutions is required when solving log equations.

Check $x = 125$:
$\log_5(125) - \log_5(125-100) = 1$
$\log_5(125) - \log_5(25) = 1$
$3 - 2 = 1$
$1 = 1$
(Checks out.)

Solution set = $\boxed{\{125\}}$.

Note (*): We could have performed base-5 exponentiation on both sides:

$$5^{\log_5\left(\frac{x}{x-100}\right)} = 5^1 \Rightarrow \frac{x}{x-100} = 5$$