MIDTERM 3 SOLUTIONS

(CHAPTER 4)

INTRODUCTION TO TRIGONOMETRY; MATH 141 – SPRING 2026 – KUNIYUKI

150 POINTS TOTAL: 30 FOR PART 1, AND 120 FOR PART 2

PART 1: USING SCIENTIFIC CALCULATORS (30 PTS.)

1) A circle has radius 4 feet. A central angle of the circle intercepts an arc of length 7 feet along the circle. What is the radian measure of the central angle? (4 points)

Arc length $s = r\theta$

$$\left(7 \ \cancel{\text{feet}}\right) = \left(4 \ \cancel{\text{feet}}\right)\theta$$

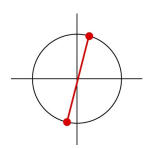
$$\frac{7}{4} = \theta$$

$$\theta = \frac{7}{4} = 1.75 \Rightarrow \boxed{\text{The central angle is } \frac{7}{4}\text{, or 1.75, radians.}}$$

2) Give the solutions for $\tan(\theta) = 3.92$, where $0 \le \theta < 2\pi$. Give your solutions in **radians** and in solution set form, and round them off to the nearest thousandth of a radian (that is, to three decimal places). (4 points)

• The **acute, Quadrant I solution** is given by: $\theta = \tan^{-1}(3.92) \approx 1.321$ in radians (the assumed measure), which is in the specified interval, $[0, 2\pi)$.

• The only other Quadrant in which **tangent is positive** in value is **Quadrant III**. Another solution is the **"brother"** or **"coreference angle"** of the first solution that is in Quadrant III and that is in $[0, 2\pi)$. This angle is about: $1.321 + \pi \approx 4.463$ [radians].



(The indicated terminal sides have the same slope, 3.92.)

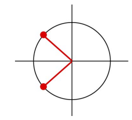
$\boxed{\text{The approximate solution set is } \{1.321,\ 4.463\}.}$

3) Give the solutions for $\cos(\theta) = -0.8$, where $0^\circ \le \theta < 360^\circ$. Give your solutions in **degrees** and in solution set form, and round them off to the nearest tenth of a degree (that is, to one decimal place). (6 points)

• In degrees, $\cos^{-1}(-0.8) \approx 143.1^\circ$, which is in $[0^\circ, 360^\circ)$, so this is one of our solutions. It is a **Quadrant II** angle.

• The only other Quadrant in which **cosine is negative** in value is **Quadrant III**. Another solution is the **"brother"** or **"coreference angle"** of the first solution that is in Quadrant III and that is in $[0^\circ, 360^\circ)$. This angle is about: $360^\circ - 143.1^\circ \approx 216.9^\circ$.

(The indicated intersection points have the same x-coordinate, -0.8.)

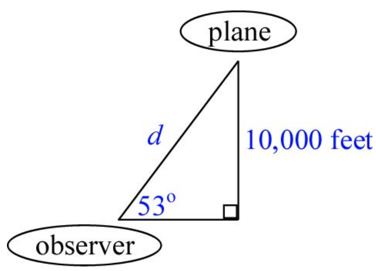


The approximate solution set is $\{143.1^\circ,\ 216.9^\circ\}$.

4) An airplane is flying at an altitude of 10,000 feet over a flat desert. In other words, its height from the ground is always 10,000 feet as far as we're concerned. It is flying on a line that takes it directly over an observer. (16 points total)

a) If the angle of elevation from the observer (specifically, the observer's shoes) to the plane is 53°, what is the distance from the observer to the plane? Round off your answer to the nearest foot.

Let d = the distance from the observer to the plane.
By SOH-CAH-TOA,



$$\sin\left(53^\circ\right) = \frac{10{,}000}{d}$$

$$d\sin\left(53^\circ\right) = 10{,}000$$

$$d = \frac{10{,}000}{\sin\left(53^\circ\right)}$$

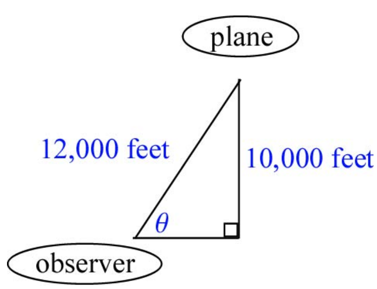
$$d \approx \boxed{12{,}521 \text{ feet}}$$

The distance from the observer to the plane is about 12,521 feet.

This seems reasonable, based on the figure.

b) Let's assume that, after some time has elapsed, the plane is now 12,000 feet away from the observer. The angle of elevation from the observer to the plane is no longer 53°, as it was in part a). Find the new angle of elevation. Round off your answer to the nearest tenth of a degree.

Let θ = the angle of elevation from the observer to the plane.
By SOH-CAH-TOA,



$$\sin(\theta) = \frac{10{,}000}{12{,}000} \text{ or } \frac{5}{6} \quad (\theta \text{ is acute})$$

$$\theta = \sin^{-1}\left(\frac{5}{6}\right) \quad (\text{but convert to degrees})$$

$$\theta \approx \boxed{56.4^\circ}$$

The new angle of elevation is about 56.4°.

This seems reasonable, based on the figure.

PART 2: NO CALCULATORS ALLOWED! (120 POINTS)

5) Fill out the table below. Rationalize denominators and simplify wherever appropriate. You do not have to show work. (28 points total)

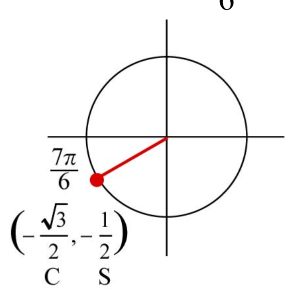
θ	$\sin(\theta)$	$\cos(\theta)$	$\tan(\theta)$	$\sec(\theta)$
0	$\frac{\sqrt{0}}{2} = \mathbf{0}$	$\mathbf{1}$	$\frac{0}{1} = \mathbf{0}$	$\frac{1}{1} = \mathbf{1}$
$\frac{\pi}{6}$	$\frac{\sqrt{1}}{2} = \frac{\mathbf{1}}{\mathbf{2}}$	$\frac{\sqrt{\mathbf{3}}}{\mathbf{2}}$	$\frac{1/2}{\sqrt{3}/2} = \frac{1}{\sqrt{3}} = \frac{\sqrt{\mathbf{3}}}{\mathbf{3}}$	$\frac{2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\mathbf{2\sqrt{3}}}{\mathbf{3}}$
$\frac{\pi}{4}$	$\frac{\sqrt{\mathbf{2}}}{\mathbf{2}}$	$\frac{\sqrt{\mathbf{2}}}{\mathbf{2}}$	$\mathbf{1}$	$\frac{2}{\sqrt{2}}$ or $\frac{1}{1/\sqrt{2}} = \sqrt{\mathbf{2}}$
$\frac{\pi}{3}$	$\frac{\sqrt{\mathbf{3}}}{\mathbf{2}}$	$\frac{\mathbf{1}}{\mathbf{2}}$	$\frac{\sqrt{3}/2}{1/2} = \sqrt{\mathbf{3}}$	$\frac{1}{1/2} = \mathbf{2}$
$\frac{\pi}{2}$	$\frac{\sqrt{4}}{2} = \mathbf{1}$	$\mathbf{0}$	$\frac{1}{0}$: **und.**	$\frac{1}{0}$: **und.**
xxx	xxxxxxxxxxx	xxxxxxxxxxx	xxxxxxxxxxx	xxxxxxxxxxx
$\frac{7\pi}{6}$	$-\frac{\mathbf{1}}{\mathbf{2}}$	$-\frac{\sqrt{\mathbf{3}}}{\mathbf{2}}$	$\frac{\sqrt{\mathbf{3}}}{\mathbf{3}}$	$-\frac{\mathbf{2\sqrt{3}}}{\mathbf{3}}$
$\frac{3\pi}{2}$	$\mathbf{-1}$	$\mathbf{0}$	$\frac{-1}{0}$: **und.**	$\frac{1}{0}$: **und.**

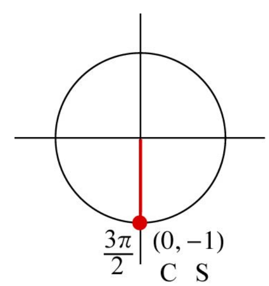
For the special angles from 0 to $\frac{\pi}{2}$:

Cofunction IDs: **Reverse** the $\sin(\theta)$ column to get the $\cos(\theta)$ column.

Quotient ID: $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$. **Reciprocal** ID: $\sec(\theta) = \frac{1}{\cos(\theta)}$.

Use the unit circle for $\frac{7\pi}{6}$ (on the **left**) and $\frac{3\pi}{2}$ (on the **right**):





Reference angle: $\frac{\pi}{6}$

Quadrant III

By ASTC, $\tan(\theta)$ and $\cot(\theta)$ are **positive** in value, while the other four are **negative**.

6) Convert 80° into radians. (3 points). $80^\circ = \left(80^\circ\right)\left(\frac{\pi}{180^\circ}\right) = \boxed{\frac{4\pi}{9}}$

7) Assuming $\tan(\theta) = 5$, find $\cot(-\theta)$. (3 points)

$$\cot(-\theta) = -\cot(\theta) \text{ (by Odd Property)} = -\frac{1}{\tan(\theta)} \text{ (by Reciprocal ID)} = \boxed{-\frac{1}{5}}$$

8) Assuming $\cos(\theta) = \frac{4}{5}$, find $\sec(-\theta)$. (3 points)

$$\sec(-\theta) = \sec(\theta) \text{ (by Even Property)} = \frac{1}{\cos(\theta)} \text{ (by Reciprocal ID)} = \frac{1}{4/5} = \boxed{\frac{5}{4}}$$

9) Complete the Identities. Fill out the table below so that, for each row, the left side is equivalent to the right side, based on the type of identity (ID) given in the last column. (8 points total)

Left Side	**Right Side**	**Type of Identity (ID)**
$\cot(\theta)$	$\frac{\cos(\theta)}{\sin(\theta)}$	Quotient ID
$\tan^2(\theta)+1$	$\sec^2(\theta)$	Pythagorean ID
$1+\cot^2(\theta)$	$\csc^2(\theta)$	Pythagorean ID
$\sec\left(\frac{\pi}{2}-\theta\right)$	$\csc(\theta)$	Cofunction ID

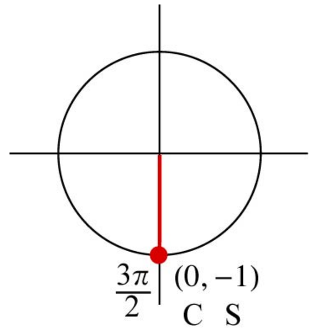
10) Fill out the table below. Use interval form (the one with parentheses and/or brackets) except where indicated. You do not have to show work. (24 points)

$f(x)$	Domain	Range
$\cos(x)$	$(-\infty, \infty)$	$[-1, 1]$
$\tan(x)$	Use set-builder form. $\left\{x \in \mathbb{R} \,\middle|\, x \neq \frac{\pi}{2} + \pi n \ (n \in \mathbb{Z})\right\}$	$(-\infty, \infty)$
$\cot(x)$	Use set-builder form. $\left\{x \in \mathbb{R} \,\middle|\, x \neq \pi n \ (n \in \mathbb{Z})\right\}$	$(-\infty, \infty)$
$\sec(x)$	Use set-builder form. $\left\{x \in \mathbb{R} \,\middle|\, x \neq \frac{\pi}{2} + \pi n \ (n \in \mathbb{Z})\right\}$	$(-\infty, -1] \cup [1, \infty)$
$\sin^{-1}(x)$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\cos^{-1}(x)$	$[-1, 1]$	$[0, \pi]$

11) Evaluate $\cot\left(\frac{7\pi}{2}\right)$. (3 points)

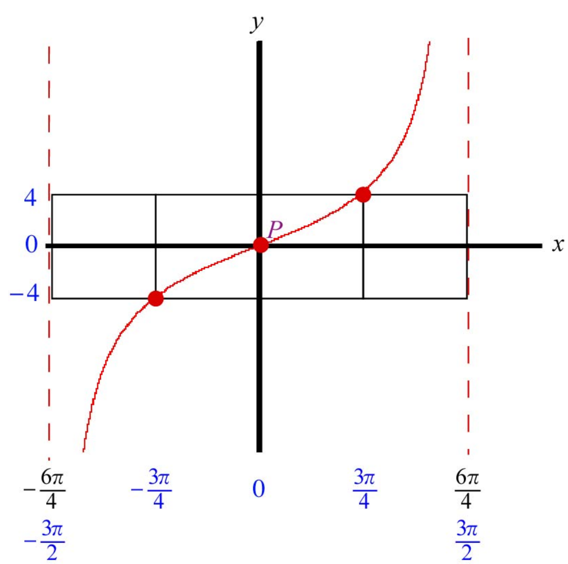
A "nice" **coterminal** angle for $\frac{7\pi}{2}$ is given by: $\frac{7\pi}{2} - 2\pi = \frac{7\pi}{2} - \frac{4\pi}{2} = \frac{3\pi}{2}$.

$\tan\left(\frac{3\pi}{2}\right)$ is **undefined**, as is $\tan\left(\frac{7\pi}{2}\right)$. Therefore, $\cot\left(\frac{7\pi}{2}\right) = \cot\left(\frac{3\pi}{2}\right) = \boxed{0}$.



12) Graph one cycle of $y = 4\tan\left(\frac{x}{3}\right)$.

Simplify and clearly label all key x- and y-coordinates next to each corresponding grid line. Superimpose the x- and y-axes. If you do not use the frame, make sure you provide all required information. (10 points)



Observe: $b = \frac{1}{3} > 0$

Pivot, P: $(0, 0)$; at frame center for tangent

Cycle shape: for tangent, $a = 4 > 0$ ("rising snake")

"**A**mplitude of Frame" (half-height of frame): $|a| = |4| = 4$

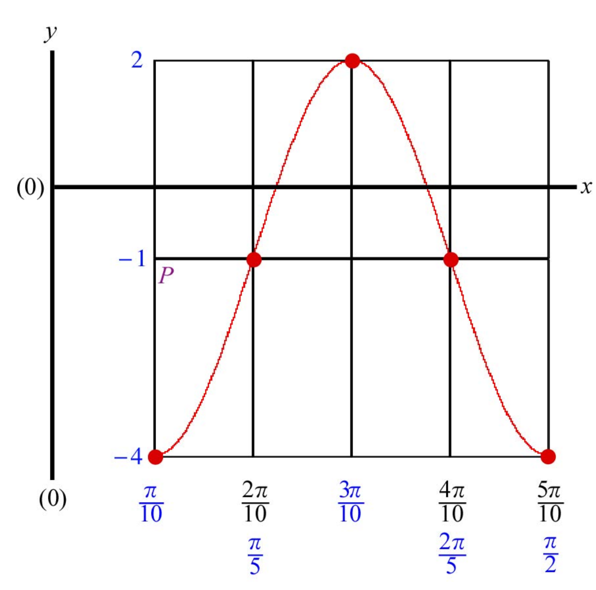
Period: $\frac{\pi}{b} = \frac{\pi}{1/3} = 3\pi$

Increment: $\frac{1}{4}(\text{Period}) = \frac{1}{4}(3\pi) = \frac{3\pi}{4}$

Axes: Remember them!

13) Graph one cycle of $y=-3\cos\left(5x-\dfrac{\pi}{2}\right)-1$.

Simplify and clearly label all key x- and y-coordinates next to each corresponding grid line. Superimpose the x- and y-axes. If you do not use the frame, make sure you provide all required information. (16 points)



Observe: $b=5>0$

Pivot, *P*: $(p,d)=\left(\dfrac{\pi}{10},-1\right)$;

see below for p; at frame's left-center

Cycle shape: for cosine, $a=-3<0$ ("hill")

Amplitude: $|a|=|-3|=3$

Period: $\dfrac{2\pi}{b}=\dfrac{2\pi}{5}$

Increment: $\dfrac{1}{4}(\text{Period})=\dfrac{1}{\cancel{4}_2}\left(\dfrac{\cancel{2}\pi}{5}\right)=\dfrac{\pi}{10}$

Axes: Remember them!

Find p, the phase shift:

Factoring method:

$$y=-3\cos\left(5x-\frac{\pi}{2}\right)-1$$

$$y=-3\cos\left[5\left(x-\underbrace{\frac{\pi}{10}}_{"p"}\right)\right]-1$$

or

Solve $5x-\dfrac{\pi}{2}=0$:

$$5x=\frac{\pi}{2}$$

$$x=\frac{\pi}{10}$$

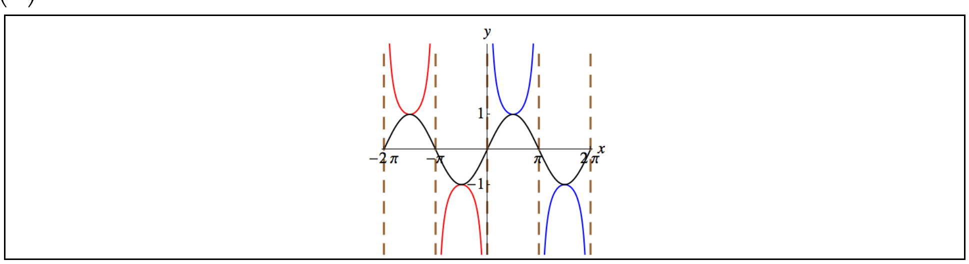
Period $=\dfrac{2\pi}{5}$, so I will accept

$$p=\frac{\pi}{10}+\frac{2\pi}{5}n\ \left(n\in\mathbb{Z}\right).$$

The phase shift $p=\dfrac{\pi}{10}$. (There is no alternative that is closer to 0.)

14) The graph of $y=\sin(x)$ appears in the box below. Graph two cycles of the graph of $y=\csc(x)$ in the box below by using the graph of $y=\sin(x)$ as a guide. Draw vertical asymptotes as dashed lines where appropriate. (5 points)

$y=\csc(x)$ is graphed in red and blue, with vertical asymptotes (VAs) dashed in brown.



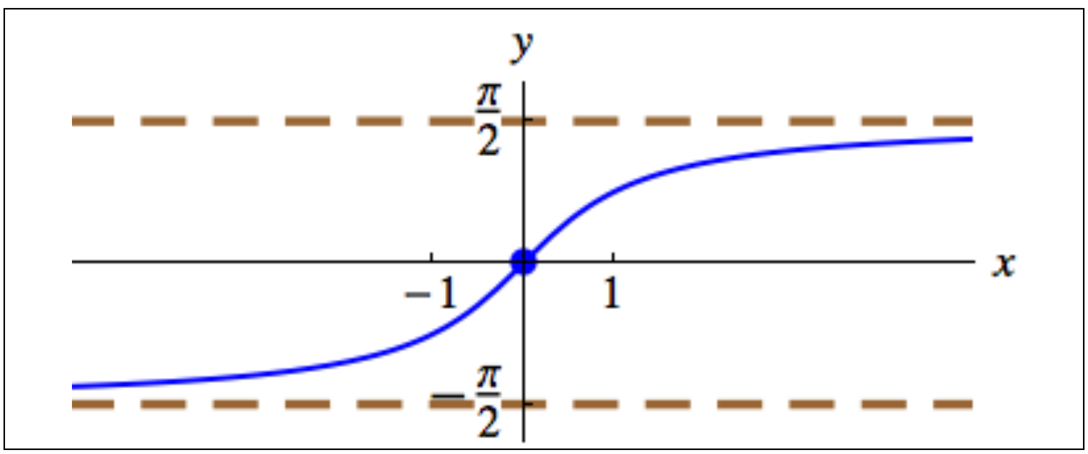
15) Evaluate $\arccos\left(-\frac{1}{2}\right)$, also written as $\cos^{-1}\left(-\frac{1}{2}\right)$. (2 points)

$\arccos\left(-\frac{1}{2}\right)=\boxed{\frac{2\pi}{3}}$, because $\cos\left(\frac{2\pi}{3}\right)=-\frac{1}{2}$, and $\frac{2\pi}{3}$ is in the arccosine function's **range**, $[0,\pi]$.

16) Evaluate $\arcsin\left(\sin\left(\frac{4\pi}{3}\right)\right)$, also written as $\sin^{-1}\left(\sin\left(\frac{4\pi}{3}\right)\right)$. (2 points)

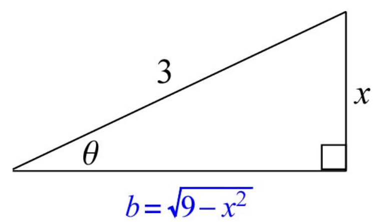
$\arcsin\left(\sin\left(\frac{4\pi}{3}\right)\right) = \arcsin\left(-\frac{\sqrt{3}}{2}\right) = \boxed{-\frac{\pi}{3}}$. Also, $-\frac{\pi}{3}$ is the "brother" ("coreference angle") in the arcsine function's **range**, $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, that shares the same sine value $\left(-\frac{\sqrt{3}}{2}\right)$ as $\frac{4\pi}{3}$ does.

17) Graph $y=\tan^{-1}(x)$, also known as $y=\arctan(x)$, in the usual xy-plane. Draw in the x- and y-axes, and clearly indicate any asymptotes and their corresponding coordinates. (6 points)



18) Write $\sec\left(\sin^{-1}\left(\frac{x}{3}\right)\right)$, also written as $\sec\left(\arcsin\left(\frac{x}{3}\right)\right)$, as an equivalent algebraic expression, as in class. Assume x is in the domain of the expression. (7 points)

Let $\theta = \sin^{-1}\left(\frac{x}{3}\right)$, or $\sin^{-1}\left(\frac{x}{3}\right) = \theta$. Then, $\sin(\theta) = \frac{x}{3}$. We may assume θ is acute when we draw our right triangle model using SOH-CAH-TOA.



Use the **Pythagorean Theorem** to find the missing (blue) side length, b:

$$(x)^2 + (b)^2 = (3)^2$$
$$x^2 + b^2 = 9$$
$$b^2 = 9 - x^2$$
$$b = \cancel{\pm}\sqrt{9 - x^2} \quad \text{(Take the nonnegative root.)}$$

By SOH-CAH-TOA, $\cos(\theta) = \frac{\text{Adj.}}{\text{Hyp.}}$, so its reciprocal, $\sec(\theta) = \frac{\text{Hyp.}}{\text{Adj.}}$.

$$\sec\left(\sin^{-1}\left(\frac{x}{3}\right)\right) = \sec(\theta) = \boxed{\frac{3}{\sqrt{9 - x^2}}} \quad \left(\text{Warning: } \sqrt{9 - x^2} \neq 3 - x\text{, usually}\right)$$