

MIDTERM 4 SOLUTIONS

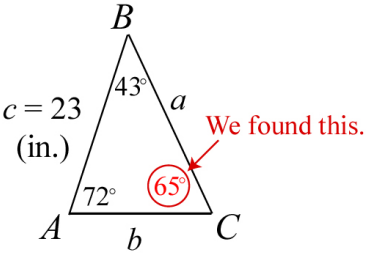
(CHAPTERS 5 AND 6: ANALYTIC & MISC. TRIGONOMETRY)

MATH 141 – SPRING 2026 – KUNIYUKI

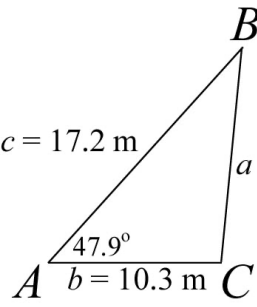
150 POINTS TOTAL: 47 FOR PART 1, AND 103 FOR PART 2

PART 1: USING SCIENTIFIC CALCULATORS (47 PTS.)

- 1) Find the length of Side b for the triangle below using the Law of Sines. Round off your answer to the nearest tenth (that is, to one decimal place) of an inch. (8 points)

	<p>Find Angle C: $C = 180^\circ - 72^\circ - 43^\circ = 65^\circ$.</p> <p>Use the Law of Sines. We now know B, C, and c, and we want to find b.</p> $\frac{b}{\sin(B)} = \frac{c}{\sin(C)}$ $\frac{b}{\sin(43^\circ)} = \frac{23}{\sin(65^\circ)}$ $b = \frac{23\sin(43^\circ)}{\sin(65^\circ)}$ $b \approx \boxed{17.3 \text{ inches}}$ <p>This makes sense, because b is shorter than c, and Angle B is smaller than Angle C. Remember that larger angles face ("eat") longer sides in a triangle. Note: $a \approx 24.1$ in.</p>
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- 2) A tilting flagpole is represented by line segment \overline{BC} in the figure below. An observer stands at point A . The observer's shoes are 10.3 meters from (and are level with) the base of the pole and are 17.2 meters from the top of the pole. The angle of elevation from the observer's shoes to the top of the pole is 47.9° . Find the length of the flagpole, which is given by a , the length of \overline{BC} . Use the Law of Cosines. Round off your answer to the nearest tenth (that is, to one decimal place) of a meter. Note: Angle C is obtuse, not right. (7 points)

	$a^2 = b^2 + c^2 - 2bc \cos(A) \quad (\text{Law of Cosines}) \Rightarrow$ $a = \sqrt{b^2 + c^2 - 2bc \cos(A)} \quad (\text{Nonnegative root only; no } \pm \text{ sign.})$ $= \sqrt{(10.3)^2 + (17.2)^2 - 2(10.3)(17.2)\cos(47.9^\circ)} \quad (\approx \sqrt{164.38})$ $\approx \boxed{12.8 \text{ m}}$ <p>The flagpole is about 12.8 m long. (Note: $B \approx 36.6^\circ$ and $C \approx 95.5^\circ$.)</p>
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For the rest of Part 1, assume that x and y are scaled in meters.

- 3) For parts a) and b) below, consider the vector \mathbf{v} , where $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j}$.
(11 points total)

- a) Find the direction angle of \mathbf{v} . Round off your answer to the nearest tenth (that is, to one decimal place) of a degree. (4 points)

Let θ be an appropriate choice for the direction angle. Then, $\tan(\theta) = \frac{b}{a} = \frac{3}{2}$.

The position vector for \mathbf{v} does lie in **Quadrant I**, so it is appropriate to take:

$$\theta = \tan^{-1}\left(\frac{3}{2}\right) \text{ (though we want this in degrees)} \approx \boxed{56.3^\circ}.$$

- b) Find the vector of magnitude 10 meters in the direction of the vector \mathbf{v} . Write the vector in $\langle x, y \rangle$ component form. Give an **exact** answer; do **not** approximate! Rationalize denominators in your answer. (7 points)

• Find the length (or magnitude) of \mathbf{v} : $\|\mathbf{v}\| = \sqrt{(2)^2 + (3)^2} = \sqrt{13}$ m.

• The unit vector \mathbf{u} in the direction of \mathbf{v} is:

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\langle 2, 3 \rangle}{\sqrt{13}} = \left\langle \frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}} \right\rangle = \left\langle \frac{2\sqrt{13}}{13}, \frac{3\sqrt{13}}{13} \right\rangle.$$

• The vector \mathbf{w} of magnitude 10 in the direction of both \mathbf{v} and \mathbf{u} is:

$$\mathbf{w} = 10\mathbf{u} = 10 \left\langle \frac{2\sqrt{13}}{13}, \frac{3\sqrt{13}}{13} \right\rangle = \boxed{\left\langle \frac{20\sqrt{13}}{13} \text{ m}, \frac{30\sqrt{13}}{13} \text{ m} \right\rangle}$$

- 4) Find the $\langle x, y \rangle$ component form of the vector \mathbf{v} that has magnitude 4 meters and direction angle 51° . Round off the x and y components to the nearest hundredth (that is, to two decimal places) of a meter. (5 points)

$$\begin{aligned} \mathbf{v} &= \langle x, y \rangle = \langle \|\mathbf{v}\| \cos(\theta), \|\mathbf{v}\| \sin(\theta) \rangle \text{ (or think: } \langle r \cos(\theta), r \sin(\theta) \rangle) \\ &= \langle 4 \cos(51^\circ), 4 \sin(51^\circ) \rangle \approx \boxed{\langle 2.52 \text{ m}, 3.11 \text{ m} \rangle} \end{aligned}$$

5) Assume that \mathbf{v} and \mathbf{w} are two vectors in the real plane. (6 points total)

a) $(\mathbf{v} \bullet \mathbf{w})\|\mathbf{w}\|$ is ... (Box in one:) a scalar a vector (Neither)

$\mathbf{v} \bullet \mathbf{w}$ is a scalar, and $\|\mathbf{w}\|$ is a scalar, so their product is a scalar. “SS = S.”

b) $(\mathbf{v} \bullet \mathbf{w}) \bullet \mathbf{v}$ is ... (Box in one:) a scalar a vector (Neither)

$\mathbf{v} \bullet \mathbf{w}$ is a scalar, and the dot product of a scalar and a vector (“S • V”) is undefined.

c) $(\mathbf{v} \bullet \mathbf{w})\mathbf{w}$ is ... (Box in one:) a scalar a vector (Neither)

$\mathbf{v} \bullet \mathbf{w}$ is a scalar, so $(\mathbf{v} \bullet \mathbf{w})\mathbf{w}$, which is a scalar times a vector, is a vector.
“SV = V.” Scalar multiplication of a vector yields a vector.

6) Consider the vectors \mathbf{v} and \mathbf{w} , where $\mathbf{v} = \langle 4, -3 \rangle$ and $\mathbf{w} = \langle -2, 1 \rangle$. Find the angle between \mathbf{v} and \mathbf{w} using the formula given in class. Round off your answer to the nearest tenth (that is, to one decimal place) of a degree. Assume distance is in meters. (10 points)

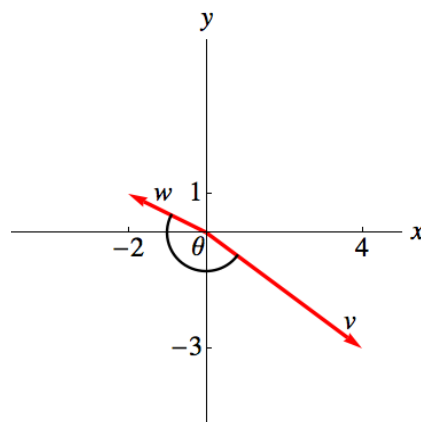
Let θ represent the desired angle.

$$\begin{aligned} \cos(\theta) &= \frac{\mathbf{v} \bullet \mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} = \frac{\langle 4, -3 \rangle \bullet \langle -2, 1 \rangle}{\|\langle 4, -3 \rangle\|\|\langle -2, 1 \rangle\|} = \frac{(4)(-2) + (-3)(1)}{\sqrt{(4)^2 + (-3)^2} \sqrt{(-2)^2 + (1)^2}} \\ &= \frac{-8 - 3}{\sqrt{25}\sqrt{5}} = \frac{-11}{5\sqrt{5}} = -\frac{11\sqrt{5}}{25} (\approx -0.98387) \Rightarrow \end{aligned}$$

$$\theta = \cos^{-1}\left(-\frac{11\sqrt{5}}{25}\right) [\text{in degrees}] \approx \boxed{169.7^\circ}$$

(Put your calculator in **degree** mode when pressing the \cos^{-1} button. Ignore “Quadrant issues” here, because the range of the inverse cosine (or arccosine) function is $[0, \pi]$.)

Our result is consistent with the sketch below; we make sure the axes are scaled the same:



PART 2: NO CALCULATORS ALLOWED! (103 PTS.)

7) Complete the Identities. You do not have to show work. (16 points total)

Left Side	Right Side	Type of Identity (ID)
$\sin(u + v)$	$\sin(u)\cos(v) + \cos(u)\sin(v)$	Sum ID
$\cos(u + v)$	$\cos(u)\cos(v) - \sin(u)\sin(v)$	Sum ID
$\tan(u + v)$	$\frac{\tan(u) + \tan(v)}{1 - \tan(u)\tan(v)}$	Sum ID
$\sin(u - v)$	$\sin(u)\cos(v) - \cos(u)\sin(v)$	Difference ID
$\sin(2u)$	$2\sin(u)\cos(u)$	Double-Angle ID
$\sin^2(u)$	$\frac{1 - \cos(2u)}{2}$ or $\frac{1}{2} - \frac{1}{2}\cos(2u)$	Power-Reducing ID (PRI)
$\cos^2(u)$	$\frac{1 + \cos(2u)}{2}$ or $\frac{1}{2} + \frac{1}{2}\cos(2u)$	Power-Reducing ID (PRI)
$\cos\left(\frac{\theta}{2}\right)$	$\pm \sqrt{\frac{1 + \cos(\theta)}{2}}$ (Choose the sign appropriately.)	Half-Angle ID

8) Write the three versions of the Double-Angle Identities for $\cos(2u)$ that we discussed in class; these were listed on “More Trig Identities.” (6 points)

Version 1:	$\cos(2u) = \cos^2(u) - \sin^2(u)$
Version 2:	$\cos(2u) = 1 - 2\sin^2(u)$
Version 3:	$\cos(2u) = 2\cos^2(u) - 1$

9) Write the three Half-Angle Identities for $\tan\left(\frac{\theta}{2}\right)$, as given in class. (3 points)

$\tan\left(\frac{\theta}{2}\right) = \pm \sqrt{\frac{1 - \cos(\theta)}{1 + \cos(\theta)}}$, $\tan\left(\frac{\theta}{2}\right) = \frac{1 - \cos(\theta)}{\sin(\theta)}$, and $\tan\left(\frac{\theta}{2}\right) = \frac{\sin(\theta)}{1 + \cos(\theta)}$
(Choose sign appropriately.)

10) Verify the identity: $\frac{\sin(2x)}{1 + \tan^2(x)} = 2 \sin(x) \cos^3(x)$. (6 points)

$$\begin{aligned}\frac{\sin(2x)}{1 + \tan^2(x)} &= \frac{2 \sin(x) \cos(x)}{\sec^2(x)} \quad \left(\begin{array}{l} \leftarrow \text{by a Double-Angle Identity} \\ \leftarrow \text{by a Pythagorean Identity} \end{array} \right) \\ &= 2 \sin(x) \cos(x) \cdot \frac{1}{\sec^2(x)} \\ &= 2 \sin(x) \cos(x) \cdot \cos^2(x) \quad (\text{by a Reciprocal Identity}) \\ &= 2 \sin(x) \cos^3(x)\end{aligned}$$

Q.E.D.

11) Verify the identity: $\frac{1 + \csc(-\theta)}{\cot(\theta)} = \tan(\theta) - \sec(\theta)$. (8 points)

$$\begin{aligned}\frac{1 + \csc(-\theta)}{\cot(\theta)} &= \frac{1 - \csc(\theta)}{\cot(\theta)} \quad [\text{by Odd Property for } \csc(\theta)] \\ &= \frac{1}{\cot(\theta)} - \frac{\csc(\theta)}{\cot(\theta)} \quad [\text{by Splitting through the Numerator}] \\ &= \tan(\theta) - \frac{\frac{1}{\sin(\theta)}}{\frac{\cos(\theta)}{\sin(\theta)}} \quad [\text{by Reciprocal and Quotient Identities}] \\ &= \tan(\theta) - \frac{1}{\underbrace{\sin(\theta)}_{(i)}} \cdot \frac{\overbrace{\sin(\theta)}^{(i)}}{\cos(\theta)} \\ &= \tan(\theta) - \frac{1}{\cos(\theta)} \\ &= \tan(\theta) - \sec(\theta) \quad [\text{by Reciprocal Identities}]\end{aligned}$$

Q.E.D.

- 12) Use the trigonometric substitution $x = 7 \tan(\theta)$ to rewrite the algebraic expression $\sqrt{x^2 + 49}$ as a trigonometric expression in θ , where θ is acute. Show all work, as in class, and simplify. (7 points)

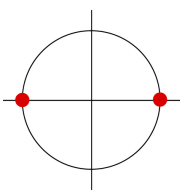
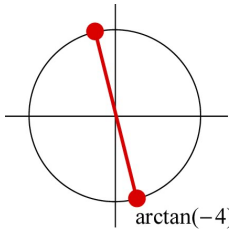
$$\begin{aligned}\sqrt{x^2 + 49} &= \sqrt{[7 \tan(\theta)]^2 + 49} = \sqrt{49 \tan^2(\theta) + 49} \\ &= \sqrt{49[\tan^2(\theta) + 1]} = \sqrt{49} \cdot \sqrt{\tan^2(\theta) + 1} = 7\sqrt{\sec^2(\theta)} \text{ (by a Pythagorean ID)} \\ &= 7|\sec(\theta)| \text{ (because } \sqrt{x^2} = |x|) = \boxed{7\sec(\theta)}\end{aligned}$$

Since θ is acute, we know $\sec(\theta) > 0$, and so $|\sec(\theta)| = \sec(\theta)$.

- 13) Find all real solutions of the equation: $\tan^2(x) + 4 \tan(x) = 0$.
Give **exact** answers; do **not** approximate! (8 points)

$$\begin{aligned}\tan^2(x) + 4 \tan(x) &= 0 \text{ [Think: Solve } u^2 + 4u = 0.] \\ [\tan(x)][\tan(x) + 4] &= 0 \text{ [Think: Solve } u(u + 4) = 0.] \end{aligned}$$

Use the **Zero Factor Property (ZFP)**; set each factor equal to 0 and solve.

<p><u>First factor:</u></p> $\tan(x) = 0$ <p>“Tangent is slope.”</p>  $x = \pi n \quad (n \in \mathbb{Z})$	<p><u>Second factor:</u></p> $\tan(x) + 4 = 0$ $\tan(x) = -4$ <p>This does have real solutions. Tangent values are negative in Quadrants II and IV.</p>  $x = \arctan(-4) + \pi n \quad (n \in \mathbb{Z})$
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The **solution set** is: $\boxed{\{x \in \mathbb{R} \mid x = \pi n, \text{ or } x = \arctan(-4) + \pi n \quad (n \in \mathbb{Z})\}}$.

14) Consider the equation: $2\sin(4x) - \sqrt{3} = 0$. (18 points total)

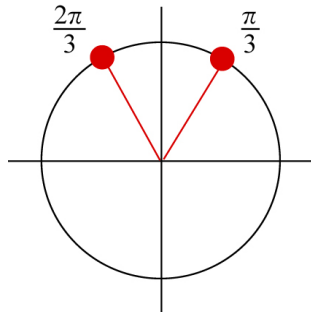
a) Find all real solutions of the equation: $2\sin(4x) - \sqrt{3} = 0$.

$$2\sin(4x) - \sqrt{3} = 0$$

$$\sin(4x) = \frac{\sqrt{3}}{2}$$

Let $\theta = 4x$, and solve: $\sin(\theta) = \frac{\sqrt{3}}{2}$.

Sine values are positive in Quadrants I and II:



$$\theta = \frac{\pi}{3} + 2\pi n \quad (n \in \mathbb{Z})$$

$$\theta = \frac{2\pi}{3} + 2\pi n \quad (n \in \mathbb{Z})$$

$$4x = \frac{\pi}{3} + 2\pi n \quad (n \in \mathbb{Z})$$

$$4x = \frac{2\pi}{3} + 2\pi n \quad (n \in \mathbb{Z})$$

$$x = \left(\frac{1}{4}\right)\left(\frac{\pi}{3}\right) + \frac{2\pi n}{4} \quad (n \in \mathbb{Z})$$

or

$$x = \left(\frac{1}{4}\right)\left(\frac{2\pi}{3}\right) + \frac{2\pi n}{4} \quad (n \in \mathbb{Z})$$

$$x = \frac{\pi}{12} + \frac{\pi}{2}n \quad (n \in \mathbb{Z})$$

$$x = \frac{\pi}{6} + \frac{\pi}{2}n \quad (n \in \mathbb{Z})$$

The **solution set** is: $\left\{ x \in \mathbb{R} \mid x = \frac{\pi}{12} + \frac{\pi}{2}n \text{ or } x = \frac{\pi}{6} + \frac{\pi}{2}n \quad (n \in \mathbb{Z}) \right\}$.

- b) Use part a) to find the real solutions of the equation $2\sin(4x) - \sqrt{3} = 0$ in the interval $[0, 2\pi)$. You do not have to use set notation, but make sure to box in all your solutions.

Group 1: $x = \frac{\pi}{12} + \frac{\pi}{2}n$ ($n \in \mathbb{Z}$). **Start**: $\frac{\pi}{12}$ ($n=0$); **Increment**: $\frac{\pi}{2}$, or $\frac{6\pi}{12}$

(Optional)	$n < 0$	$n = 0$	$n = 1$	$n = 2$	$n = 3$	$n > 3$
Particular Solutions	(outside the interval)	$\frac{\pi}{12}$	$\frac{7\pi}{12}$	$\frac{13\pi}{12}$	$\frac{19\pi}{12}$	(outside the interval)

Group 2: $x = \frac{\pi}{6} + \frac{\pi}{2}n$ ($n \in \mathbb{Z}$). **Start**: $\frac{\pi}{6}$ ($n=0$); **Increment**: $\frac{\pi}{2}$, or $\frac{3\pi}{6}$

(Optional)	$n < 0$	$n = 0$	$n = 1$	$n = 2$	$n = 3$	$n > 3$
Particular Solutions	(outside the interval)	$\frac{\pi}{6}$	$\frac{4\pi}{6}$ $= \frac{2\pi}{3}$	$\frac{7\pi}{6}$	$\frac{10\pi}{6}$ $= \frac{5\pi}{3}$	(outside the interval)

The **solution set** is (sorting optional): $\left\{ \frac{\pi}{12}, \frac{\pi}{6}, \frac{7\pi}{12}, \frac{2\pi}{3}, \frac{13\pi}{12}, \frac{7\pi}{6}, \frac{19\pi}{12}, \frac{5\pi}{3} \right\}$.

- 15) Use **one** of the Product-to-Sum Identities below to rewrite the expression $\sin(7\theta)\cos(\theta)$. Simplify. (4 points)

$$\sin(u)\sin(v) = \frac{1}{2}[\cos(u-v) - \cos(u+v)]$$

$$\cos(u)\cos(v) = \frac{1}{2}[\cos(u-v) + \cos(u+v)]$$

$$\sin(u)\cos(v) = \frac{1}{2}[\sin(u+v) + \sin(u-v)]$$

$$\cos(u)\sin(v) = \frac{1}{2}[\sin(u+v) - \sin(u-v)]$$

Use the third identity (ID) with $u = 7\theta$ and $v = \theta$:

$$\begin{aligned} \sin(7\theta)\cos(\theta) &= \frac{1}{2}[\sin(7\theta + \theta) + \sin(7\theta - \theta)] \\ &= \frac{1}{2}[\sin(8\theta) + \sin(6\theta)] \end{aligned}$$

Warning: This is **not** equivalent to $\frac{1}{2}[\sin(14\theta)]$.

- 16) Simplify as an exact value: $\cos(50^\circ)\cos(20^\circ) + \sin(50^\circ)\sin(20^\circ)$. (4 points)

Use the **Difference Identity (ID)** “right-to-left”:

$$\cos(u-v) = \cos(u)\cos(v) + \sin(u)\sin(v), \text{ where } u = 50^\circ \text{ and } v = 20^\circ.$$

$$\cos(50^\circ)\cos(20^\circ) + \sin(50^\circ)\sin(20^\circ) = \cos(50^\circ - 20^\circ) = \cos(30^\circ) = \boxed{\frac{\sqrt{3}}{2}}$$

- 17) Use a Double-Angle Identity to simplify $10\sin(x)\cos(x)$. (4 points)

$$10\sin(x)\cos(x) = 5[2\sin(x)\cos(x)] = \boxed{5\sin(2x)}$$

- 18) Simplify: $\sin^2(15^\circ)$. Hint: Use a Power-Reducing ID (PRI). Make sure your final answer is not a compound fraction. (5 points)

$$\sin^2(u) = \frac{1 - \cos(2u)}{2}. \text{ We want to use this PRI with } u = 15^\circ \text{ and } 2u = 30^\circ.$$

$$\sin^2(15^\circ) = \frac{1 - \cos(30^\circ)}{2} = \frac{1 - \frac{\sqrt{3}}{2}}{2} = \frac{\left(1 - \frac{\sqrt{3}}{2}\right) \cdot 2}{2 \cdot 2} = \boxed{\frac{2 - \sqrt{3}}{4}}$$

- 19) Use the Power-Reducing Identities (PRIs) to rewrite $\sin^4(x)$ using only constants, signs, and the first power of cosine expressions (and no other powers). Fill in the blanks below with real numbers in simplified form. Show all work! (11 points)

$$\sin^4(x) = \boxed{\frac{3}{8}} - \boxed{\frac{1}{2}}\cos(2x) + \boxed{\frac{1}{8}}\cos(4x)$$

$$\sin^4(x) = [\sin^2(x)]^2 = \left[\frac{1 - \cos(2x)}{2}\right]^2 \text{ (by PRI)} = \frac{[1 - \cos(2x)]^2}{4}$$

$$= \frac{1}{4}[1 - \cos(2x)]^2 = \frac{1}{4}[1 - 2\cos(2x) + \cos^2(2x)]$$

$$= \frac{1}{4}\left[1 - 2\cos(2x) + \frac{1 + \cos(4x)}{2}\right] \text{ (by PRI)} = \frac{1}{4}\left[1 - 2\cos(2x) + \frac{1}{2} + \frac{1}{2}\cos(4x)\right]$$

$$= \frac{1}{4}\left[\frac{3}{2} - 2\cos(2x) + \frac{1}{2}\cos(4x)\right] = \frac{3}{8} - \frac{1}{2}\cos(2x) + \frac{1}{8}\cos(4x)$$

- 20) A triangle has one side of length a and another side of length b , and Angle C is the included angle between the two sides. Give the formula we gave in class for the area of this triangle in terms of C , a , and b . (3 points)

$$\text{Area} = \frac{1}{2} ab \sin(C)$$