

QUIZ ON CHAPTERS 1 AND 2 - SOLUTIONS

REVIEW / LIMITS AND CONTINUITY; MATH 150 – FALL 2016 – KUNIYUKI
105 POINTS TOTAL, BUT 100 POINTS = 100%

- 1) If $f(x) = x^{2/3}$, then the graph of $y = f(x)$ in the usual (Cartesian) xy -plane is symmetric about what? (Box in one:) (2 points)

the x -axis

the y -axis

the origin

none of these

This is because f is an even function. Observe that $\text{Dom}(f) = \mathbb{R}$. $\forall x \in \mathbb{R}$,

$$\begin{aligned} f(x) &= x^{2/3} = \sqrt[3]{x^2} \Rightarrow \\ f(-x) &= \sqrt[3]{(-x)^2} = \sqrt[3]{x^2} = x^{2/3} = f(x) \end{aligned}$$

- 2) Fill in the blanks. Find rules for functions f and g so that $(f \circ g)(x) = f(g(x)) = \sin(x^3)$. (We are decomposing a composite function.)

$$\boxed{g(x) = x^3, \quad f(u) = \sin(u)}$$

(Do not let f or g be the identity function.) (2 points)

There are different possible answers, but the above would be a reasonable choice.

- 3) Complete the Identities. Fill out the table below so that, for each row, the left side is equivalent to the right side, based on the type of identity (ID) given in the last column. (10 points total; 2 points each)

Left Side	Right Side	Type of Identity (ID)
$\sin(u + v)$	$\sin(u)\cos(v) + \cos(u)\sin(v)$	Sum ID
$\cos(u - v)$	$\cos(u)\cos(v) + \sin(u)\sin(v)$	Difference ID
$\sin(2u)$	$2\sin(u)\cos(u)$	Double-Angle ID
$\cos(2u)$	$\cos^2(u) - \sin^2(u)$, $1 - 2\sin^2(u)$, or $2\cos^2(u) - 1$	Double-Angle ID (write any one of our three versions)
$\cos^2(u)$	$\frac{1 + \cos(2u)}{2}$ or $\frac{1}{2} + \frac{1}{2}\cos(2u)$	Power-Reducing ID (PRI)

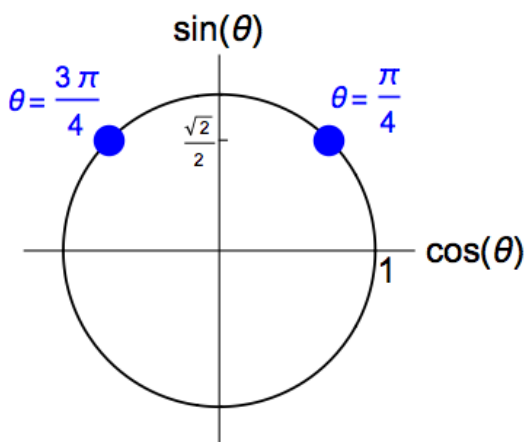
- 4) Fill out the table below. Use interval form (the form using parentheses and/or brackets) except where indicated. You do not have to show work. (6 points)

$f(x)$	Domain	Range
$\sin(x)$	$(-\infty, \infty)$	$[-1, 1]$
$\cot(x)$	Use set-builder form. $\left\{x \in \mathbb{R} \mid x \neq \pi n \ (n \in \mathbb{Z})\right\}$	$(-\infty, \infty)$
$\sec(x)$	Use set-builder form. $\left\{x \in \mathbb{R} \mid x \neq \frac{\pi}{2} + \pi n \ (n \in \mathbb{Z})\right\}$	$(-\infty, -1] \cup [1, \infty)$

- 5) Find all real solutions of $2 \sin(4x) - \sqrt{2} = 0$ in radians. (8 points)

$$2 \sin(4x) - \sqrt{2} = 0$$

$$\sin(\underbrace{4x}_{=\theta}) = \frac{\sqrt{2}}{2}$$



$$\theta = \frac{\pi}{4} + 2\pi n$$

$$\theta = \frac{3\pi}{4} + 2\pi n$$

$$4x = \frac{\pi}{4} + 2\pi n$$

$$4x = \frac{3\pi}{4} + 2\pi n$$

or

$$x = \frac{\pi}{16} + \frac{2\pi n}{4}$$

$$x = \frac{3\pi}{16} + \frac{2\pi n}{4}$$

$$x = \frac{\pi}{16} + \frac{\pi n}{2}$$

$$x = \frac{3\pi}{16} + \frac{\pi n}{2}$$

where $n \in \mathbb{Z}$.

The solution set is:

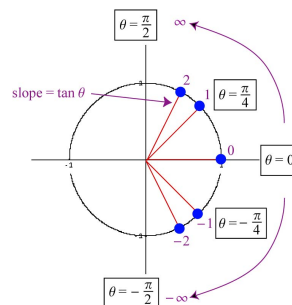
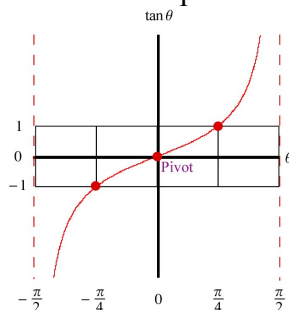
$$\left\{x \in \mathbb{R} \mid x = \frac{\pi}{16} + \frac{\pi n}{2} \text{ or } x = \frac{3\pi}{16} + \frac{\pi n}{2} \ (n \in \mathbb{Z})\right\}$$

If a limit does not exist, and ∞ and $-\infty$ are inappropriate, write “DNE.”

6) Evaluate the following limits. **Box in your final answers.** (15 points total)

a) $\lim_{\theta \rightarrow \left(\frac{\pi}{2}\right)^-} \tan(\theta)$ **Answer only is fine.** (2 points)

Answer: $\boxed{\infty}$. Consider the fundamental cycle of $y = \tan(\theta)$ in the figure on the left, or consider the interpretation of tangent as slope on the right.



b) $\lim_{r \rightarrow 7} \frac{\frac{1}{r} - \frac{1}{7}}{r - 7}$ **Show all work.** (7 points)

$$\begin{aligned} & \lim_{r \rightarrow 7} \frac{\frac{1}{r} - \frac{1}{7}}{r - 7} \quad \left(\text{Limit Form } \frac{0}{0} \right) \\ &= \lim_{r \rightarrow 7} \frac{\left(\frac{1}{r} - \frac{1}{7} \right)}{(r - 7)} \cdot \frac{7r}{7r} \quad \left(\text{Use the LCD, } 7r, \text{ to resolve the compound fraction.} \right) \\ &= \lim_{r \rightarrow 7} \frac{\overset{(-1)}{\cancel{(7-r)}}}{7r \underset{(1)}{\cancel{(r-7)}}} = \lim_{r \rightarrow 7} \left(-\frac{1}{7r} \right) = -\frac{1}{7(7)} = \boxed{-\frac{1}{49}} \end{aligned}$$

c) $\lim_{x \rightarrow 0} x^4 \sin\left(\frac{1}{\sqrt[3]{x}}\right)$ **Show all work, as we have done in class.** (6 pts.)

Answer: $\boxed{0}$. Prove this using the Sandwich / Squeeze Theorem:

$$-1 \leq \sin\left(\frac{1}{\sqrt[3]{x}}\right) \leq 1 \quad (\forall x \neq 0)$$

Observe that $x^4 > 0$, $\forall x \neq 0$. Multiply all three parts by x^4 .

$$\text{As } x \rightarrow 0, \underbrace{-x^4}_{\rightarrow 0} \leq \underbrace{x^4 \sin\left(\frac{1}{\sqrt[3]{x}}\right)}_{\substack{\text{So, } \rightarrow 0 \\ \text{by Sandwich/} \\ \text{Squeeze Thm.}}} \leq \underbrace{x^4}_{\rightarrow 0} \quad (\forall x \neq 0)$$

More precisely: $\lim_{x \rightarrow 0} (-x^4) = 0$, and $\lim_{x \rightarrow 0} x^4 = 0$.

Therefore, by the Sandwich / Squeeze Theorem, $\lim_{x \rightarrow 0} x^4 \sin\left(\frac{1}{\sqrt[3]{x}}\right) = 0$.

7) Consider $f(x) = \frac{x+1}{x^2-3x-10}$ and the graph of $y = f(x)$ in the usual xy -plane in parts a) through e). (16 points total)

a) Find $\lim_{x \rightarrow -2^-} f(x)$. **Show all work, as in class.** (6 points)

$$\begin{aligned}\lim_{x \rightarrow -2^-} f(x) &= \lim_{x \rightarrow -2^-} \frac{x+1}{x^2-3x-10} \\ &= \lim_{x \rightarrow -2^-} \frac{x+1}{\underbrace{(x+2)}_{\rightarrow 0^-} \underbrace{(x-5)}_{\rightarrow -7}} \quad \left(\begin{array}{c} \rightarrow (-2)+1=-1 \\ \rightarrow 0^+ \end{array} \right) \quad \left(\text{Limit Form: } \frac{-1}{0^+} \right) \\ &= \boxed{-\infty}\end{aligned}$$

b) Find $\lim_{x \rightarrow \infty} f(x)$. **Answer only is fine.** (2 points)

$\lim_{x \rightarrow \infty} f(x) = \boxed{0}$. This is because $\frac{x+1}{x^2-3x-10}$ is a rational expression written as a quotient of nonzero polynomials, and the degree of the numerator (1) is less than the degree of the denominator (2). Think: **“Bottom-heavy.”**

c) Find the equation(s) of the vertical asymptote(s) (VAs) of the graph of $y = f(x)$. **Answer only is fine.** (4 points)

$\frac{x+1}{x^2-3x-10}$, or $\frac{x+1}{(x+2)(x-5)}$, is a simplified rational expression, so the real

zeros of the denominator yield the VAs. The VAs are at $\boxed{x = -2 \text{ and } x = 5}$.

Note: Part a) proves that there is a VA at $x = -2$.

d) Find the equation of the horizontal asymptote (HA) of the graph of $y = f(x)$. **Answer only is fine.** (2 points)

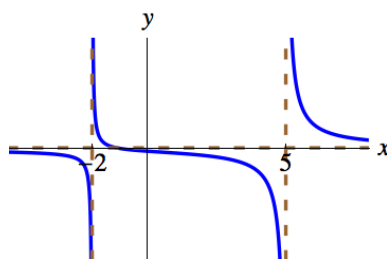
The HA is at: $\boxed{y = 0}$.

This is because $\lim_{x \rightarrow \infty} f(x) = 0$; see Part b). Also, $\lim_{x \rightarrow -\infty} f(x) = 0$.

e) How many holes does the graph of $y = f(x)$ have? **Answer only.** (2 pts.)

$\boxed{\text{None.}}$ This is because the $(x+2)$ and $(x-5)$ factors are **not** canceled (divided) out of the denominator.

Here is the graph of $y = f(x)$:



8) Consider $f(x) = \frac{5x^4}{2x^4 - 18x^2}$ and its graph in the usual xy -plane. (16 pts. total)

a) Find $\lim_{x \rightarrow \infty} f(x)$. **Give a rigorous solution (no short cuts!).** (5 points)

We are analyzing the “long-run” limit of a rational function as $x \rightarrow \infty$.

Method 1 (Division Method)

Divide numerator and denominator by x^4 , the highest power of x in the denom.

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{5x^4}{2x^4 - 18x^2} = \lim_{x \rightarrow \infty} \frac{\frac{5x^4}{x^4}}{\frac{2x^4}{x^4} - \frac{18x^2}{x^4}} = \lim_{x \rightarrow \infty} \frac{5}{2 - \underbrace{\left(\frac{18}{x^2}\right)}_{\rightarrow 0}} = \boxed{\frac{5}{2}}$$

Method 2 (Factoring Method)

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{5x^4}{2x^4 - 18x^2} = \lim_{x \rightarrow \infty} \frac{5x^4}{2x^4 \left(1 - \frac{18x^2}{2x^4}\right)} = \lim_{x \rightarrow \infty} \frac{5x^4}{2x^4 \left[1 - \underbrace{\left(\frac{9}{x^2}\right)}_{\rightarrow 0}\right]} = \lim_{x \rightarrow \infty} \frac{5x^4}{2x^4 \underbrace{\left[1 - \left(\frac{9}{x^2}\right)\right]}_{\rightarrow 1}}$$

$$= \lim_{x \rightarrow \infty} \frac{5 \cancel{x^4}^{(i)}}{2 \cancel{x^4}^{(i)}} \cdot \underbrace{\left[\frac{1}{1 - \frac{9}{x^2}}\right]}_{\rightarrow 1} = \boxed{\frac{5}{2}}$$

b) Find the equation(s) of the vertical asymptote(s) (VAs) of the graph of $y = f(x)$. **You should show some work.** (6 points)

$$f(x) = \frac{5x^4}{2x^4 - 18x^2} = \frac{5x^4}{2x^2(x^2 - 9)} = \frac{5 \overbrace{\cancel{x^4}^{(x^2)}}^{(x^2)}}{2 \cancel{x^2}^{(i)}(x+3)(x-3)} = \frac{5x^2}{2(x+3)(x-3)} \quad (x \neq 0)$$

$\frac{5x^2}{2(x+3)(x-3)}$ is a simplified rational expression, so the VAs correspond to the

real zeros of the denominator. The VAs are at $\boxed{x = -3 \text{ and } x = 3}$.

c) Find the equation of the horizontal asymptote (HA) of the graph of $y = f(x)$. **Answer only is fine.** (2 points)

The HA is at $\boxed{y = \frac{5}{2}}$, because $\lim_{x \rightarrow \infty} f(x) = \frac{5}{2}$; see 8a. Also, $\lim_{x \rightarrow -\infty} f(x) = \frac{5}{2}$.

“Super short cut”: $\frac{5x^4}{2x^4 - 18x^2}$ is a rational expression written as a quotient of nonzero polynomials of **equal degree** (4). The **ratio of the leading coefficients** of the polynomials is $\frac{5}{2}$, so that is the “long-run” limit value of f in both directions.

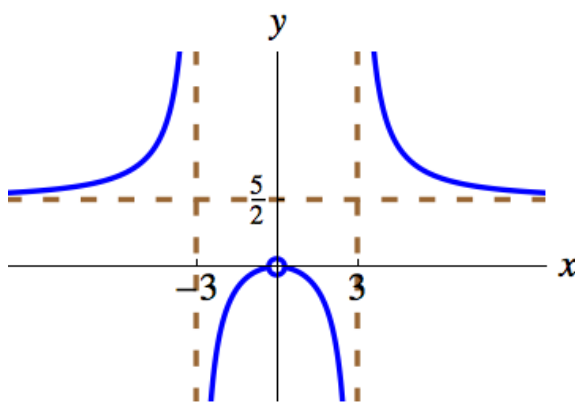
d) Where is the hole on the graph of $y = f(x)$? Write the x - and y -coordinates in (x, y) form. **Answer only is fine.** (3 points)

Answer: $\boxed{(0, 0)}$.

- $x = 0$ corresponds to a hole, because there are x -factors of the denominator of the original rational expression $\frac{5x^4}{2x^4 - 18x^2}$, but all of them are canceled (divided) out of the denominator in the simplification process.
- The other variable factors of the denominator, $(x + 3)$ and $(x - 3)$, are not canceled (divided) out. They correspond to VAs, not holes.
- The hole is at $\left(0, \lim_{x \rightarrow 0} f(x)\right)$. Using the simplified form of $f(x)$ from Part b),

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \frac{5x^2}{2(x+3)(x-3)} = \frac{5(0)^2}{2(0+3)(0-3)} = 0.$$

Here is the graph of $y = f(x)$:



9) Write a precise ε - δ definition of $\lim_{x \rightarrow a} f(x) = L$ ($a, L \in \mathbb{R}$).

Assume f is defined on a punctured neighborhood of a . (7 points)

$\forall \quad \varepsilon > 0, \quad \exists \quad \delta > 0$ such that if $0 < x - a < \delta$, then $ f(x) - L < \varepsilon$.
<div style="display: flex; justify-content: space-around; font-size: small;"> <div> $\underbrace{\forall}_{\text{For every / any / all}}$ </div> <div> $\underbrace{\exists}_{\text{there exists}}$ </div> </div>

Alternatively: ... then $f(x)$ is in $(L - \varepsilon, L + \varepsilon)$

10) Let $f(x) = \begin{cases} \sqrt{x+2}, & x \neq 7 \\ 4, & x = 7 \end{cases}$. (6 points total)

a) What is $\lim_{x \rightarrow 7} f(x)$? (3 points)

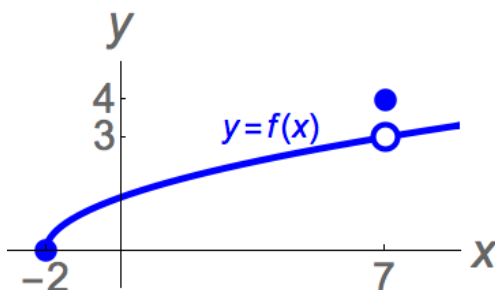
The only relevant rule “immediately” to the left and to the right of $x = 7$ (that is, in a punctured neighborhood of $x = 7$) is $f(x) = \sqrt{x+2}$. (The 4 is irrelevant for now.) $\lim_{x \rightarrow 7} f(x) = \lim_{x \rightarrow 7} \sqrt{x+2} = \sqrt{(7)+2} = \sqrt{9} = \boxed{3}$.

b) Classify the discontinuity of f at $x = 7$. Box in one: (3 points)

Infinite discontinuity Jump discontinuity Removable discontinuity

$\lim_{x \rightarrow 7} f(x) = 3$, a real number, yet $f(7) = 4 \neq 3$. There is a hole at $(7, 3)$.

The discontinuity would be “removed” if $f(7)$ were redefined to be 3.



11) Let $f(x) = \frac{|x-4|}{x-4}$. (9 points total; 3 points each)

$$f(x) = \begin{cases} \frac{x-4}{x-4} = 1, & x > 4 \\ -\frac{(x-4)}{x-4} = -1, & x < 4 \end{cases}$$

a) Find $\lim_{x \rightarrow 4^-} f(x)$.

$\lim_{x \rightarrow 4^-} f(x) = \boxed{-1}$. As $x \rightarrow 4^-$, $x < 4$, and the rule $f(x) = -1$ applies.

b) Find $\lim_{x \rightarrow 4^+} f(x)$.

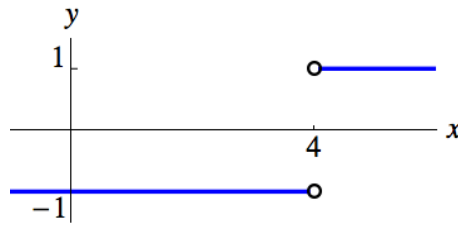
$\lim_{x \rightarrow 4^+} f(x) = \boxed{1}$. As $x \rightarrow 4^+$, $x > 4$, and the rule $f(x) = 1$ applies.

c) Classify the discontinuity of f at $x = 4$. Box in one:

Infinite discontinuity Jump discontinuity Removable discontinuity

The above left-hand and right-hand limits exist but are different ($-1 \neq 1$).

Here is the graph of $y = f(x)$:



- 12) Let $f(x) = \frac{\sqrt{x-3}}{x-5}$. What is the domain of f ? Write your answer in interval form (the form using parentheses and/or brackets). Note: f is continuous on the domain interval(s). (6 points)

- The numerator, $\sqrt{x-3}$, is defined on $[3, \infty)$ because:
 $\sqrt{x-3}$ is real $\Leftrightarrow x-3 \geq 0 \Leftrightarrow x \geq 3$.
- The denominator, $x-5$, is defined on \mathbb{R} , but we require $x \neq 5$ to avoid a 0 denominator.
- Note: By the Algebra of Continuity Theorems, f is continuous on $(3, 5), (5, \infty)$.
 $\lim_{x \rightarrow 3^+} f(x) = f(3)$; both sides equal 0. Thus, f is right-continuous at 3.



$$\text{Dom}(f) = [3, 5) \cup (5, \infty)$$

- 13) (2 points). True or False: If f is a polynomial function on \mathbb{R} such that $f(2) = -7$ and $f(3) = 9$, then there exists a real number c in the interval $[2, 3]$ such that $f(c) = 0$. Box in one: ☒ True ☐ False

f is polynomial on \mathbb{R} and hence continuous on \mathbb{R} , in particular on the interval $[2, 3]$.
 $0 \in [-7, 9]$. By the Intermediate Value Theorem (IVT), f has a real zero in $[2, 3]$.