## QUIZ ON CHAPTERS 1 AND 2 - SOLUTIONS

REVIEW / LIMITS AND CONTINUITY; MATH 150 - SPRING 2017 - KUNIYUKI 105 POINTS TOTAL, BUT 100 POINTS $=\mathbf{1 0 0 \%}$

1) For a), b), and c) below, box in the correct answer. ( 6 points total; 2 pts. each)
a) Let $f(x)=x^{4}+\cos (x)$. The function $f$ is $\ldots$ even odd neither $\operatorname{Dom}(f)=\mathbb{R} . \forall x \in \mathbb{R}, f(-x)=(-x)^{4}+\cos (-x)=x^{4}+\cos (x)=f(x)$.
b) Let $g(x)=\sqrt[3]{x}+\sin (x)$. The function $g$ is $\ldots$ even odd neither $\operatorname{Dom}(g)=\mathbb{R} . \forall x \in \mathbb{R}, g(-x)=\sqrt[3]{-x}+\sin (-x)=-\sqrt[3]{x}-\sin (x)=-g(x)$.
c) Let $h(x)=x^{5}-x+1$. The function $h$ is $\ldots$ even odd neither. A counterexample to $h$ being even or odd: $h(2)=31$, but $h(-2)=-29$. $h(2)$ and $h(-2)$ are neither equal nor opposite.
2) Fill in the blanks. Find rules for functions $f$ and $g$ so that $(f \circ g)(x)=f(g(x))=\frac{1}{x^{2}+x}$. (We are decomposing a composite function.)

$$
g(x)=x^{2}+x, \quad f(u)=\frac{1}{u}
$$

(Do not let $f$ or $g$ be the identity function.) ( 2 points)
There are different possible answers, but the above would be a reasonable choice.
3) Complete the Identities. Fill out the table below so that, for each row, the left side is equivalent to the right side, based on the type of identity (ID) given in the last column. (8 points total; 2 points each)

| Left Side | Right Side | Type of Identity (ID) |
| :---: | :---: | :---: |
| $1+\tan ^{2}(u)$ | $\sec ^{2}(u)$ | Pythagorean ID |
| $\cos (u+v)$ | $\cos (u) \cos (v)-\sin (u) \sin (v)$ | Sum ID |
| $\sin (u-v)$ | $\sin (u) \cos (v)-\cos (u) \sin (v)$ | Difference ID |
| $\cos ^{2}(u)$ | $\frac{1+\cos (2 u)}{2}$ or $\frac{1}{2}+\frac{1}{2} \cos (2 u)$ | Power-Reducing ID (PRI) |

4) Write any two of the three different versions of the Double-Angle Identity (ID) for $\cos (2 u)$ that were listed in Chapter 1. (4 points)

Any two of these three:

$$
\cos (2 u)=\cos ^{2}(u)-\sin ^{2}(u) ; \quad \cos (2 u)=1-2 \sin ^{2}(u) ; \quad \cos (2 u)=2 \cos ^{2}(u)-1
$$

The Pythagorean Identity $\sin ^{2}(u)+\cos ^{2}(u)=1$ is used to get from the first to the others.
5) Verify the identity $\frac{\sin (2 x)}{\tan (x)}=2 \cos ^{2}(x)$ using the Chapter 1 identities. (5 points)

$$
\frac{\sin (2 x)}{\tan (x)}=\frac{2 \sin (x) \cos (x)}{\frac{\sin (x)}{\cos (x)}}\left[\frac{\leftarrow \text { Double-Angle ID }}{\leftarrow \text { Quotient ID }}\right]=2 \sin (x) \cos (x)\left(\frac{\cos (x)}{\left.\frac{\sin (x)}{(1)}\right)}\right)=2 \cos ^{2}(x)
$$

Q.E.D.
6) Fill out the table below. Use interval form (the form using parentheses and/or brackets) except where indicated. You do not have to show work. (6 points)
$\left.\left.\begin{array}{|c|c|c|}\hline f(x) & \text { Domain } & \text { Range } \\ \hline \cos (x) & (-\infty, \infty) & {[-1,1]} \\ \hline \tan (x) & \begin{array}{c}\text { Use set-builder form. } \\ \end{array} \quad\left\{x \in \mathbb{R} \left\lvert\, x \neq \frac{\pi}{2}+\pi n(n \in \mathbb{Z})\right.\right\}\end{array}\right](-\infty, \infty)\right\}$
7) In parts a) through i), consider the function $f$, where $f(x)=\frac{x^{2}-4}{3 x^{2}-5 x-2}$, and the graph of $y=f(x)$ in the usual $x y$-plane. (28 points total)
a) Find $\lim _{x \rightarrow 2} f(x)$. Show all rigorous work, as in class. (7 points)

$$
\lim _{x \rightarrow 2} f(x)=\lim _{x \rightarrow 2} \frac{x^{2}-4}{3 x^{2}-5 x-2}\left(\text { Limit Form } \frac{0}{0}\right)=\lim _{x \rightarrow 2} \frac{(x+2)(x-2)}{(3 x+1)(x-2)}
$$

$$
\begin{equation*}
=\lim _{x \rightarrow 2} \frac{x+2}{3 x+1}=\frac{(2)+2}{3(2)+1}=\frac{4}{7} \tag{1}
\end{equation*}
$$

b) Find $\lim _{x \rightarrow \infty} f(x)$. Show all rigorous work, as in class. (5 points)

Use the Division Method to find a "long-run" limit as $x \rightarrow \infty$. Divide the numerator and the denominator by $x^{2}$ (or by $x$ for the simplified form on the right), the highest power of $x$ in the denominator.

$$
\begin{aligned}
\lim _{x \rightarrow \infty} f(x) & =\lim _{x \rightarrow \infty} \frac{x^{2}-4}{3 x^{2}-5 x-2} \quad \text { or } \quad \lim _{x \rightarrow \infty} \frac{x+2}{3 x+1} \quad \text { (from Part a)) } \\
& =\lim _{x \rightarrow \infty} \frac{\frac{x^{2}}{x^{2}}-\frac{4}{x^{2}}}{\frac{3 x^{2}}{x^{2}}-\frac{5 x}{x^{2}}-\frac{2}{x^{2}}} \quad=\lim _{x \rightarrow \infty} \frac{\frac{x}{x}+\frac{2}{x}}{\frac{3 x}{x}+\frac{1}{x}} \\
& =\lim _{x \rightarrow \infty} \frac{1-\frac{4}{x^{2}}}{3-\underbrace{\frac{5}{x}-\underbrace{\frac{2}{x^{2}}}_{\rightarrow 0}}_{\rightarrow 0}} \quad=\lim _{x \rightarrow \infty} \frac{1+\frac{2}{x}}{3+\frac{1}{x}} \\
& =\underbrace{\frac{1}{x}}_{\rightarrow 0} \quad \text { (This is the ratio of the given leading coefficients; see f).) }
\end{aligned}
$$

c) Which of the following is true? Box in one: (2 points)

$$
\text { i. } f \text { is continuous at } x=-2 \text {. }
$$

ii. $f$ has a removable discontinuity at $x=-2$.
iii. $f$ has a jump discontinuity at $x=-2$.
iv. $f$ has an infinite discontinuity at $x=-2$.
$f$ is a rational function, and $-2 \in \operatorname{Dom}(f)$, so $f$ is continuous at -2 .
d) Which of the following is true? Box in one: ( 2 points)
i. $f$ is continuous at $x=-\frac{1}{3}$.
ii. $f$ has a removable discontinuity at $x=-\frac{1}{3}$.
iii. $f$ has a jump discontinuity at $x=-\frac{1}{3}$.
iv. $f$ has an infinite discontinuity at $x=-\frac{1}{3}$.

- $f$ is rational, and $x=-\frac{1}{3}$ yields the limit form $\frac{\text { nonzero real number }}{0}$.
- The corresponding "problem" factor, $(3 x+1)$, is not completely canceled (divided) out in the denominator when the original $f(x)$ expression is simplified.
- $\lim _{x \rightarrow\left(-\frac{1}{3}\right)^{+}} f(x)=\lim _{x \rightarrow\left(-\frac{1}{3}\right)^{+}} \frac{x+2}{3 x+1}\left(\right.$ Limit Form $\left.\frac{5 / 3}{0^{+}}\right)=\infty$.
- $\lim _{x \rightarrow\left(-\frac{1}{3}\right)^{-}} f(x)=\lim _{x \rightarrow\left(-\frac{1}{3}\right)^{-}} \frac{x+2}{3 x+1}\left(\right.$ Limit Form $\left.\frac{5 / 3}{0^{-}}\right)=-\infty$.
e) Which of the following is true? Box in one: (2 points)
i. $f$ is continuous at $x=2$.
ii. $f$ has a removable discontinuity at $x=2$.
iii. $f$ has a jump discontinuity at $x=2$.
iv. $f$ has an infinite discontinuity at $x=2$.
- In Part a), we determined that $\lim _{x \rightarrow 2} f(x)$ exists.
- However, $f(2)$ is undefined $(2 \notin \operatorname{Dom}(f))$ because $(x-2)$ is a factor of the denominator of $\frac{x^{2}-4}{3 x^{2}-5 x-2}$.
- Therefore, $f$ has a removable discontinuity at $x=2$.
f) What is the horizontal asymptote of the graph of $y=f(x)$ ? Write its equation. Answer only on $f$ ) is fine. (3 points)
$f$ is a rational function, so its graph has at most one horizontal asymptote (HA).
From Part b), $\lim _{x \rightarrow \infty} f(x)=\frac{1}{3}$, so the sole HA has equation: $y=\frac{1}{3}$.
Note: For both original and simplified expressions for $f(x)$, the numerator and denominator have equal degrees, and the ratio of their leading coefficients is $\frac{1}{3}$.
g) Where is the hole on the graph of $y=f(x)$ ? Use $(x, y)$ form to write its coordinates. Answer only on g) is fine. (3 points)

From Part a), $\lim _{x \rightarrow 2} f(x)=\frac{4}{7} . f$ is discontinuous at $x=2$ (it is undefined there), yet $\lim _{x \rightarrow 2} f(x)$ exists, so there is a hole at: $\left(2, \frac{4}{7}\right)$.

- Observe that the $(x-2)$ factor is completely canceled (divided) out in the denominator when the original $f(x)$ expression is simplified.
h) What is the $x$-intercept of the graph of $y=f(x)$ ? (2 points)

Set $y=0$. The real zeros of $f$ are those of $\frac{x^{2}-4}{3 x^{2}-5 x-2}$, or $\frac{x+2}{3 x+1}(x \neq 2)$ :

$$
\frac{x+2}{3 x+1}=0 \quad(x \neq 2) \Leftrightarrow x+2=0 \quad(x \neq 2) \Leftrightarrow x=-2
$$

The $x$-intercept is at $(-2,0)$.
i) What is the $y$-intercept of the graph of $y=f(x)$ ? (2 points)

Set $x=0 \Rightarrow y=f(0)=\left[\frac{x+2}{3 x+1}\right]_{x=0}=\frac{(0)+2}{3(0)+1}=\frac{2}{1}=2$
The $y$-intercept is at $(0,2)$.

The graph of $y=f(x)$ is below. Note the horizontal asymptote (HA) at $y=\frac{1}{3}$, the vertical asymptote (VA) at $x=-\frac{1}{3}$, and the hole at $\left(2, \frac{4}{7}\right)$. There is continuity at $x=-2$; in fact, the $x$-intercept is at $(-2,0)$. The $y$-intercept is at $(0,2)$.

8) Let $f(x)=\frac{3 x^{4}+x^{2}-1}{4 x^{10}-x^{5}}$. What is the equation of the horizontal asymptote of the graph of $y=f(x)$ in the usual $x y$-plane? (3 points)

The sole horizontal asymptote (HA) is given by $y=0$, because $\lim _{x \rightarrow \infty} \frac{3 x^{4}+x^{2}-1}{4 x^{10}-x^{5}}=0$.
Alternatively, $\lim _{x \rightarrow-\infty} \frac{3 x^{4}+x^{2}-1}{4 x^{10}-x^{5}}=0$. Note that $\frac{3 x^{4}+x^{2}-1}{4 x^{10}-x^{5}}$ is proper and rational, and we are taking its "long-run" limits. (By "proper," we mean that the numerator's degree, 4, is less than the denominator's degree, 10. Think: "Bottom-heavy.")
9) Find the following limits. Box in your final answers. (22 points total)
a) $\lim _{r \rightarrow 11} \frac{\sqrt{r-2}-3}{11-r}$. Show all work, as in class. (10 points)

We have a $\frac{0}{0}$ limit form. We will rationalize the numerator.

$$
\begin{align*}
& \lim _{r \rightarrow 11} \frac{\sqrt{r-2}-3}{11-r}=\lim _{r \rightarrow 11} \frac{(\sqrt{r-2}-3)}{(11-r)} \cdot \frac{(\sqrt{r-2}+3)}{(\sqrt{r-2}+3)}(\sqrt{r-2} \text { is defined near } r=11 .) \\
& =\lim _{r \rightarrow 11} \frac{(\sqrt{r-2})^{2}-(3)^{2}}{(11-r)(\sqrt{r-2}+3)}=\lim _{r \rightarrow 11} \frac{(r-2)-9}{(11-r)(\sqrt{r-2}+3)}=\lim _{r \rightarrow 11} \frac{r-11}{(11-r)(\sqrt{r-2}+3)} \tag{1}
\end{align*}
$$

[Observe: $(r-11)$ and $(11-r)$ are opposites. Their quotient is -1 .]

$$
=\lim _{r \rightarrow 11}\left(-\frac{1}{\sqrt{r-2}+3}\right)=-\frac{1}{\sqrt{11-2}+3}=-\frac{1}{\sqrt{9}+3}=-\frac{1}{3+3}=-\frac{1}{6}
$$

b) $\lim _{x \rightarrow \infty} \frac{\cos (\sqrt[3]{x})}{\sqrt[3]{x}}$. Show all work, as in class. (6 points) $\lim _{x \rightarrow \infty} \frac{\cos (\sqrt[3]{x})}{\sqrt[3]{x}}=0$. Prove this using the Sandwich / Squeeze Theorem:

$$
-1 \leq \cos (\sqrt[3]{x}) \leq 1 \quad(\forall x \in \mathbb{R})
$$

Observe that $\sqrt[3]{x}>0, \forall x>0$. Divide all three parts by $\sqrt[3]{x}$.

$$
\text { As } x \rightarrow \infty, \underbrace{-\frac{1}{\sqrt[3]{x}}}_{\rightarrow 0} \leq \underbrace{\frac{\cos (\sqrt[3]{x})}{\sqrt[3]{x}}}_{\substack{\text { So, } \rightarrow 0 \\ \text { byandwich } \\ \text { Squezeze Thm. }}} \leq \underbrace{\frac{1}{\sqrt[3]{x}}}_{\rightarrow 0} \quad(\forall x>0)
$$

More precisely:

$$
\lim _{x \rightarrow \infty} \frac{1}{\sqrt[3]{x}}=0 \text { and } \lim _{x \rightarrow \infty}-\frac{1}{\sqrt[3]{x}}=0 \text {, so by the Squeeze Theorem, } \lim _{x \rightarrow \infty} \frac{\cos (\sqrt[3]{x})}{\sqrt[3]{x}}=0 .
$$

c) $\lim _{x \rightarrow 3^{+}} \frac{x-6}{x^{2}-x-6}$. Show all work, as in class. (6 points)

$$
\lim _{x \rightarrow 3^{+}} \frac{x-6}{x^{2}-x-6}=\lim _{x \rightarrow 3^{+}} \underbrace{\frac{x-6}{(x-3)} \underbrace{(x+2)}_{\rightarrow 5}}_{\rightarrow 0^{+}}\left(\text {Limit Form } \frac{-3}{0^{+}} \frac{(\leftarrow 3-6)}{(\leftarrow \infty}\right.
$$

10) (2 points). True or False: It is possible that for a function $f, \lim _{x \rightarrow a} f(x)$ exists but $f(a)$ does not (meaning $f(a)$ is undefined). Box in one: True False
For example, see Section 2.1, Example 7; here, let's use the letter $f$ instead of $g$.
Let $f(x)=x+3(x \neq 3) . \lim _{x \rightarrow 3} f(x)=6$ (the limit exists), yet $f(3)$ is undefined.

11) Write a precise $\varepsilon-\delta$ definition of $\lim _{x \rightarrow a} f(x)=L(a, L \in \mathbb{R})$.

Assume $f$ is defined on a punctured neighborhood of $a$. (7 points)


Alternatively: ... then $f(x)$ is in $(L-\varepsilon, L+\varepsilon)$
12) Let $f(x)=\left\{\begin{array}{ll}3 x-4, & x \leq 2 \\ \frac{2}{x}, & x>2\end{array}\right.$ (6 points total)
a) Evaluate $\lim _{x \rightarrow 2^{+}} f(x)$.(2 points)

Since we are letting $x$ approach 2 from the right, the case $x>2$ is the only relevant case here, and the rule $f(x)=\frac{2}{x}$ is the only relevant rule here.

$$
\lim _{x \rightarrow 2^{+}} f(x)=\lim _{x \rightarrow 2^{+}} \frac{2}{x}=\frac{2}{2}=1
$$

b) Evaluate $\lim _{x \rightarrow 2^{-}} f(x)$. (2 points)

Since we are letting $x$ approach 2 from the left, the case $x \leq 2$ is the only relevant case here, and the rule $f(x)=3 x-4$ is the only relevant rule here.

$$
\lim _{x \rightarrow 2^{-}} f(x)=\lim _{x \rightarrow 2^{-}}(3 x-4)=3(2)-4=2
$$

c) Which of the following is true? Box in one: (2 points)
i. $f$ is continuous at $x=2$.
ii. $f$ has a removable discontinuity at $x=2$.
iii. $f$ has a jump discontinuity at $x=2$.
iv. $f$ has an infinite discontinuity at $x=2$.

The right-hand limit from a) and the left-hand limit from b) as $x \rightarrow 2$ both exist, but they are not equal. Therefore, $f$ has a jump discontinuity at $x=2$.
13) Let $g(t)=\frac{1}{\sqrt{t-4}}+\sqrt[3]{t-7}$. What is the domain of $g$ ? Write your answer in interval form (the form using parentheses and/or brackets). Note: $g$ is continuous on the domain interval(s). (4 points)

- The denominator, $\sqrt{t-4}$, is real and nonzero $\Leftrightarrow t-4>0 \Leftrightarrow t>4$.
- $\sqrt[3]{t-7}$ is real everywhere on $\mathbb{R}$.

$\operatorname{Dom}(g)=(4, \infty)$

14) (2 points). True or False: If $f$ is a polynomial function on $\mathbb{R}$ such that $f(1)=10$ and $f(7)=20$, then the equation $f(x)=15$ has a real solution. Box in one: True False
$f$ is polynomial on $\mathbb{R}$ and hence continuous on $\mathbb{R}$, in particular on the interval $[1,7]$. $15 \in[10,20]$. By the Intermediate Value Theorem (IVT), $f(x)=15$ has a real solution in $[1,7]$.
