QUIZ ON CHAPTER 3 - SOLUTIONS

DERIVATIVES; MATH 150 – FALL 2016 – KUNIYUKI
105 POINTS TOTAL, BUT 100 POINTS = 100%

1) Use the limit definition of the derivative to prove that $D_x\left[\sin(x)\right]=\cos(x)$, as in class. Show all steps! (12 points)

Let $f(x)=\sin(x)$.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h} \\
&= \lim_{h\to 0}\frac{\overbrace{\sin(x)\cos(h)+\cos(x)\sin(h)}^{\text{from Sum Identity for sine}}\ -\ \sin(x)}{h} \\
&= \lim_{h\to 0}\frac{\overbrace{\left[\sin(x)\cos(h)-\sin(x)\right]}^{\text{Group terms with }\sin(x).}\ +\ \cos(x)\sin(h)}{h} \\
&= \lim_{h\to 0}\frac{\left[\sin(x)\right]\left[\cos(h)-1\right]\ +\ \cos(x)\sin(h)}{h} \\
&= \lim_{h\to 0}\left[\left[\sin(x)\right]\underbrace{\left[\frac{\cos(h)-1}{h}\right]}_{\to 0}+\left[\cos(x)\right]\underbrace{\left[\frac{\sin(h)}{h}\right]}_{\to 1}\right] \quad \text{(Form fractions in } h\text{.)} \\
&= \cos(x)
\end{aligned}$$

Q.E.D.

2) A particle moving along a coordinate line has as its position function s, where $s(t)=3t+\frac{2}{\sqrt[3]{t^2}}$, for $t\geq 1$. Position $s(t)$ is measured in meters, and time t is measured in seconds. Find the velocity of the particle at time $t=8$ (seconds). Write an exact answer with correct units. (8 points)

$$s(t) = 3t+\frac{2}{\sqrt[3]{t^2}} = 3t+2t^{-2/3} \quad\Rightarrow$$

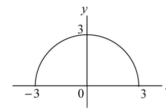
$$v(t) = s'(t) = 3+2\left(-\frac{2}{3}t^{-5/3}\right) = 3-\frac{4}{3}t^{-5/3} = 3-\frac{4}{3t^{5/3}} \quad\Rightarrow$$

$$v(8) = 3-\frac{4}{3(8)^{5/3}} = 3-\frac{4}{3\left(\sqrt[3]{8}\right)^5} = 3-\frac{4}{3(2)^5} = 3-\frac{4}{96} = 3-\frac{1}{24}$$

$$= \boxed{2\frac{23}{24}\text{ or }\frac{71}{24}\frac{\text{meters}}{\text{second}}}$$

3) If $f(x)=\sqrt{9-x^2}$, is f differentiable on the interval $[-3,3]$? Box in one:

(1 point). Yes $\boxed{\text{No}}$



The graph of $y=f(x)$ has (one-sided) vertical tangent lines at $x=\pm 3$:

Also, $f'(x)$ is undefined at $x=\pm 3$:

$$f(x) = \sqrt{9-x^2} = \left(9-x^2\right)^{1/2} \Rightarrow f'(x) = \cancel{\frac{1}{2}}\left(9-x^2\right)^{-1/2}\left(-\cancel{2}x\right) = -\frac{x}{\sqrt{9-x^2}}$$

4) Let $f(x)=x^2+x$. These are linearization problems. (11 points total)

a) Find an equation of the tangent line to the graph of $y=f(x)$ at the point on the graph (in the usual *xy*-plane) where $x=3$. Use any form. (8 points)

Find the *y*-coordinate of the point of tangency on the graph.

$$x_1=3 \Rightarrow y_1 = f(3) = (3)^2+3 = 12$$

Find the slope, *m*, of the tangent line there.

$$f'(x)=2x+1 \Rightarrow m = f'(3) = 2(3)+1 = 7$$

Point-slope form of the tangent line:

$$y-y_1=m(x-x_1) \Rightarrow$$

$$\boxed{y-12=7(x-3)}$$

Slope-intercept form:

$$y-12=7x-21$$

$$\boxed{y=7x-9}$$

or

$$y=mx+b \Rightarrow$$

$$(12)=(7)(3)+b$$

$$b=-9 \Rightarrow$$

$$\boxed{y=7x-9}$$

b) Use your tangent line from a) to give a linear approximation for the value of $f(2.85)$. Give your answer as a decimal written out to two decimal places. (3 points)

Using Point-slope form:

$$y-12=7(2.85-3)$$

$$y-12=7(-0.15)$$

$$y-12=-1.05$$

$$y=12-1.05$$

$$y=\boxed{10.95}$$

Using Slope-intercept form:

$$y = 7(2.85)-9 = \boxed{10.95}$$

Note: In fact, $f(2.85)=10.9725$. Our approximation is correct to one decimal place.

5) Find the indicated derivatives. (21 points total; 7 points each)

a) Find $\frac{d}{dr}\left(\sqrt[7]{r^6}-\frac{5}{r^3}+6\right)$. Simplify. Do not leave negative exponents in your final answer. Your final answer does not need to be a single fraction.

$$\frac{d}{dr}\left(\sqrt[7]{r^6}-\frac{5}{r^3}+6\right) = D_r\left(r^{6/7}-5r^{-3}+6\right) = \frac{6}{7}r^{\left(\frac{6}{7}-1\right)}-5(-3)r^{(-3-1)}$$

$$= \frac{6}{7}r^{-\frac{1}{7}}+15r^{-4} = \boxed{\frac{6}{7r^{1/7}}+\frac{15}{r^4}, \text{ or } \frac{6r^{27/7}+105}{7r^4}, \text{ or } \frac{3\left(2r^{27/7}+35\right)}{7r^4}}$$

b) Let $f(x)=3x^7\sin\left(x^2\right)$. Find $f'(x)$. Simplify. You do not have to factor your final answer.

Apply the Product Rule. Then, use the Generalized Trigonometric Rules for the second term.

$$f'(x)=\left[D_x\left(3x^7\right)\right]\cdot\left[\sin\left(x^2\right)\right] + \left[3x^7\right]\cdot\left[D_x\left(\sin\left(x^2\right)\right)\right]$$

$$=\left[21x^6\right]\cdot\left[\sin\left(x^2\right)\right] + \left[3x^7\right]\cdot\left[\cos\left(x^2\right)\right]\cdot\underbrace{\left[D_x\left(x^2\right)\right]}_{=(2x)}$$

$$=\boxed{21x^6\sin\left(x^2\right)+6x^8\cos\left(x^2\right), \text{ or } 3x^6\left[7\sin\left(x^2\right)+2x^2\cos\left(x^2\right)\right]}$$

c) Find $D_\alpha\left[\csc^3\left(7\alpha+\frac{\pi}{4}\right)\right]$. Simplify.

Rewrite, and then apply the Generalized Power Rule.

$$D_\alpha\left[\csc^3\left(7\alpha+\frac{\pi}{4}\right)\right]=D_\alpha\left[\csc\left(7\alpha+\frac{\pi}{4}\right)\right]^3$$

$$=3\left[\csc\left(7\alpha+\frac{\pi}{4}\right)\right]^2\cdot\left(D_\alpha\left[\csc\left(7\alpha+\frac{\pi}{4}\right)\right]\right)$$

$$=3\left[\csc\left(7\alpha+\frac{\pi}{4}\right)\right]^2\cdot\left[-\csc\left(7\alpha+\frac{\pi}{4}\right)\cot\left(7\alpha+\frac{\pi}{4}\right)\right]\cdot\underbrace{\left[D_\alpha\left(7\alpha+\frac{\pi}{4}\right)\right]}_{=(7)}$$

$$=\boxed{-21\csc^3\left(7\alpha+\frac{\pi}{4}\right)\cot\left(7\alpha+\frac{\pi}{4}\right)}$$

6) Find $D_x\left(\dfrac{7x+1}{\sqrt{3x^2+5}}\right)$. Simplify. Your final answer must have the form: $\dfrac{\text{an } x \text{ term and a constant term}}{\left(3x^2+5\right)^{\text{some exponent}}}$. (12 points)

$$D_x\left(\frac{7x+1}{\sqrt{3x^2+5}}\right) = \frac{\mathrm{Lo}\cdot \mathrm{D}(\mathrm{Hi}) - \mathrm{Hi}\cdot \mathrm{D}(\mathrm{Lo})}{(\mathrm{Lo})^2} \quad \text{(Quotient Rule)}$$

$$= \frac{\left[\sqrt{3x^2+5}\right]\cdot\left[D_x(7x+1)\right] - \left[7x+1\right]\cdot\left[D_x\left(\sqrt{3x^2+5}\right)\right]}{\left(\sqrt{3x^2+5}\right)^2}$$

$$= \frac{\left[\sqrt{3x^2+5}\right]\cdot[7] - [7x+1]\cdot\left(D_x\left[\left(3x^2+5\right)^{1/2}\right]\right)}{3x^2+5}$$

$$= \frac{\left[\sqrt{3x^2+5}\right]\cdot[7] - [7x+1]\cdot\left[\frac{1}{2}\left(3x^2+5\right)^{-1/2}\right]\left[D_x\left(3x^2+5\right)\right]}{3x^2+5} \quad \text{(Gen. Power Rule)}$$

$$= \frac{\left[\sqrt{3x^2+5}\right]\cdot[7] - [7x+1]\cdot\left[\frac{1}{2}\left(3x^2+5\right)^{-1/2}\right][6x]}{3x^2+5}$$

$$= \frac{7\sqrt{3x^2+5} - 3x[7x+1]\cdot\left[\left(3x^2+5\right)^{-1/2}\right]}{3x^2+5}$$

Let $u=\left(3x^2+5\right)$.

$$= \frac{7u^{1/2} - 3x[7x+1]\cdot\left[u^{-1/2}\right]}{u}$$

<u>Method 1</u>: Factor the numerator **before** dividing powers of u.

$$= \frac{u^{-1/2}\left[7u-3x(7x+1)\right]}{u} \quad \left(\text{Now, } \frac{u^{-1/2}}{u^1} = u^{-\frac{1}{2}-1} = u^{-\frac{3}{2}} = \frac{1}{u^{3/2}}.\right)$$

$$= \frac{7u-3x(7x+1)}{u^{3/2}} \quad \left[\text{Do } \textbf{not} \text{ divide powers of } u \text{ here. Sub back } u=\left(3x^2+5\right).\right]$$

$$= \frac{7\left(3x^2+5\right)-21x^2-3x}{\left(3x^2+5\right)^{3/2}} = \frac{\cancel{21x^2}+35-\cancel{21x^2}-3x}{\left(3x^2+5\right)^{3/2}} = \boxed{\frac{35-3x}{\left(3x^2+5\right)^{3/2}}}$$

Method 2: Rewrite as a compound (or complex) fraction.

$$= \frac{7\sqrt{u} \; - \; \dfrac{3x[7x+1]}{\sqrt{u}}}{u}$$

$$= \frac{\left(7\sqrt{u} \; - \; \dfrac{3x[7x+1]}{\sqrt{u}}\right)}{u} \cdot \frac{\sqrt{u}}{\sqrt{u}} \quad \left(\text{Distribute } \sqrt{u} \text{ to } \textbf{both} \text{ terms in red.}\right)$$

$$= \frac{7u - 3x[7x+1]}{u^{3/2}} \quad \left[\text{Do } \textbf{not} \text{ divide powers of } u \text{ here. Sub back } u = \left(3x^2+5\right).\right]$$

$$= \frac{7\left(3x^2+5\right) - 3x[7x+1]}{\left(3x^2+5\right)^{3/2}} = \frac{\cancel{21x^2} + 35 - \cancel{21x^2} - 3x}{\left(3x^2+5\right)^{3/2}} = \boxed{\frac{35-3x}{\left(3x^2+5\right)^{3/2}}}$$

7) Prove that $D_x\left[\cot(x)\right] = -\csc^2(x)$ without using the limit definition of the derivative. Among the trigonometric derivatives, you may use only the derivatives of $\sin(x)$ and $\cos(x)$ without proof. Show all work! (8 points)

$$D_x\left[\cot(x)\right]$$

$$= D_x\left[\frac{\cos(x)}{\sin(x)}\right] \quad \text{(by the Quotient Identities)}$$

$$= \frac{\text{Lo}\cdot \text{D(Hi)} \; - \; \text{Hi}\cdot \text{D(Lo)}}{(\text{Lo})^2} \quad \text{(by the Quotient Rule for Differentiation)}$$

$$= \frac{[\sin(x)]\cdot\left(D_x[\cos(x)]\right) \; - \; [\cos(x)]\cdot\left(D_x[\sin(x)]\right)}{[\sin(x)]^2}$$

$$= \frac{[\sin(x)]\cdot[-\sin(x)] \; - \; [\cos(x)]\cdot[\cos(x)]}{[\sin(x)]^2}$$

$$= \frac{-\sin^2(x) - \cos^2(x)}{\sin^2(x)}$$

$$= -\frac{\sin^2(x) + \cos^2(x)}{\sin^2(x)} \quad \left[\text{Can: } = -\left[\frac{\sin^2(x)}{\sin^2(x)} + \frac{\cos^2(x)}{\sin^2(x)}\right] = -\left[1+\cot^2(x)\right] = -\csc^2(x)\right]$$

$$= -\frac{1}{\sin^2(x)} \quad \text{(by the Pythagorean Identities)}$$

$$= -\csc^2(x) \quad \text{(by the Reciprocal Identities)}$$

Q.E.D.

8) Let $f(x) = (x^2 - 4)^3$. (10 points total)

a) Find $f'(x)$. (3 points)

$$f'(x) = \left[3(x^2-4)^2\right] \cdot \left[D_x(x^2-4)\right] \quad \text{(Generalized Power Rule)}$$

$$= \left[3(x^2-4)^2\right] \cdot [2x] = \boxed{6x(x^2-4)^2}$$

b) Find the points on the graph of $y = f(x)$ (in the usual xy-plane) at which the tangent line is horizontal. (7 points)

Solve $f'(x) = 0$ for x, since horizontal lines have slope 0.

$$6x(x^2-4)^2 = 0$$

$$x^2 - 4 = 0$$

$$x = 0 \quad \text{or} \quad x^2 = 4$$

$$x = \pm 2$$

Find the corresponding y-coordinates by evaluating f at those x-values:

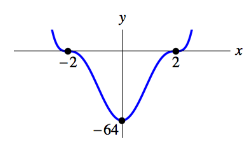
$$f(0) = \left[(0)^2 - 4\right]^3 = [-4]^3 = -64$$

$$f(x) = (x^2-4)^3 \quad \Rightarrow \quad f(2) = \left[(2)^2 - 4\right]^3 = [0]^3 = 0$$

$$f(-2) = \left[(-2)^2 - 4\right]^3 = [0]^3 = 0$$

The points we want are: $\boxed{(0, -64),\ (2, 0),\ \text{and } (-2, 0)}$.

Note: f is an even function. Here is the graph of $y = f(x)$:



9) Consider the given equation $6y + 4x^5y^2 - \cos(y) = 12$. Assume that it "determines" an implicit differentiable function f such that $y = f(x)$.

Find $\frac{dy}{dx}$ (you may use the y' notation, instead). Use implicit differentiation, as in class. (11 points)

$$6y + 4x^5y^2 - \cos(y) = 12 \quad \Rightarrow$$

$$D_x\left[6y + \underbrace{4x^5y^2}_{\text{Use the Product Rule!}} - \cos(y)\right] = D_x(12)$$

$$6y' + \left[D_x(4x^5)\right] \cdot \left[y^2\right] + \left[4x^5\right] \cdot \left[D_x(y^2)\right] + \left[\sin(y)\right]\left[y'\right] = 0$$

$$6y' + \left[20x^4\right] \cdot \left[y^2\right] + \left[4x^5\right] \cdot \left[2yy'\right] + y'\sin(y) = 0$$

$$6y' + 20x^4y^2 + 8x^5yy' + y'\sin(y) = 0$$

Now, isolate the terms with y' on one side.

$$6y' + 8x^5yy' + y'\sin(y) = -20x^4y^2$$

Factor out y' on the left side.

$$y'\left[6 + 8x^5y + \sin(y)\right] = -20x^4y^2$$

Divide to solve for y'.

$$\boxed{\frac{dy}{dx} \text{ or } y' = -\frac{20x^4y^2}{6 + 8x^5y + \sin(y)}}$$

10) Air is being pumped into a spherical balloon at the rate of 2.3 cubic feet per minute. The balloon maintains a spherical shape throughout. At what rate is the radius of the balloon changing when the radius is 30 **inches** in length?

• In your final answer, give the appropriate units, and round off your answer as a decimal to four significant digits. (Round off intermediate results to at least four significant digits.)

• Hint 1: If you forgot the "key formula" here, you can buy it from me for 2 points. You can't get negative points for this problem.

• Hint 2: One foot is equivalent to 12 inches. (11 points)

Step 1: Read the problem!

Step 2: Define variables / General diagram (optional here)

Let t = time in minutes.
Let V = the volume of the balloon [at time t].
Let r = the radius of the balloon [at time t].

Step 3: Given / Find (what?) at the instant of interest.

Given: $\dfrac{dV}{dt} = 2.3\left(\dfrac{\text{ft}^3}{\text{min}}\right)$.

Find: $\dfrac{dr}{dt}$ when $r = 30$ in.

Convert units: $(30\ \cancel{\text{in}})\left(\dfrac{1\text{ ft}}{12\ \cancel{\text{in}}}\right) = 2.5$ ft.

Step 4: Key formula. Volume $V = \dfrac{4}{3}\pi r^3$.

Step 5: Perform Implicit Differentiation on the formula in Step 4.

$$V = \frac{4}{3}\pi r^3 \quad \Rightarrow$$

$$D_t(V) = D_t\left(\frac{4}{3}\pi r^3\right)$$

$$\frac{dV}{dt} = \frac{4}{3}\pi\left(3r^2 \cdot \frac{dr}{dt}\right) \quad \text{(by the Chain Rule)}$$

$$\frac{dV}{dt} = 4\pi r^2 \cdot \frac{dr}{dt}$$

Step 6: Plug in values at the instant of interest.

Step 7: Solve for $\frac{dr}{dt}$.

$$2.3 = 4\pi(2.5)^2 \cdot \frac{dr}{dt}$$

$$2.3 = 25\pi \cdot \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{2.3}{25\pi} \quad \left(= \frac{23}{250\pi} \right) \quad (\text{Type: } 23 \div 250 \div \pi =)$$

$$\frac{dr}{dt} \approx \boxed{0.02928 \frac{\text{ft}}{\text{min}}}$$

Step 8: Conclusion.

The radius of the balloon is growing at the rate of about 0.02928 $\frac{\text{ft}}{\text{min}}$ when its radius is 30 inches.

Note on Unit Conversions:

Given: $\frac{dV}{dt} = \left(\frac{2.3 \text{ ft}^3}{\text{min}}\right)\left(\frac{12 \text{ in}}{1 \text{ ft}}\right)^3 = \left(\frac{2.3 \cancel{\text{ft}^3}}{\text{min}}\right)\left(\frac{1728 \text{ in}^3}{1 \cancel{\text{ft}^3}}\right) = 3974.4 \frac{\text{in}^3}{\text{min}}$

Find: $\frac{dr}{dt}$ when $r = 30$ in.

Step 7:

$$3974.4 = 4\pi(30)^2 \cdot \frac{dr}{dt}$$

$$3974.4 = 3600\pi \cdot \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{3974.4}{3600\pi} \quad \left(= \frac{39744}{36{,}000\pi} = \frac{138}{125\pi} \right)$$

$$\frac{dr}{dt} \approx \boxed{0.3514 \frac{\text{in}}{\text{min}}}$$

Observe:

$$\left(\frac{0.3514 \cancel{\text{in}}}{\text{min}}\right)\left(\frac{1 \text{ ft}}{12 \cancel{\text{in}}}\right) \approx 0.02928 \frac{\text{ft}}{\text{min}}, \text{ our original answer.}$$