1) Use the limit definition of the derivative to prove that \( D_x \left[ \sin(x) \right] = \cos(x) \), as in class. Show all steps! (12 points)

Let \( f(x) = \sin(x) \).

\[
\begin{align*}
    f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
    &= \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h} \\
    &= \lim_{h \to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h} \\
    &= \lim_{h \to 0} \left( \sin(x) \cos(h) - \sin(x) \right) + \cos(x)\sin(h) \\
    &= \lim_{h \to 0} \left[ \sin(x) \left( \cos(h) - 1 \right) \right] + \cos(x)\sin(h) \\
    &= \lim_{h \to 0} \left[ \sin(x) \left( \frac{\cos(h) - 1}{h} \right) \right] + \cos(x)\left( \frac{\sin(h)}{h} \right) \\
    &= \cos(x)
\end{align*}
\]

Q.E.D.

2) A particle moving along a coordinate line has as its position function \( s \), where \( s(t) = 3t + \frac{2}{\sqrt[3]{t^2}} \), for \( t \geq 1 \). Position \( s(t) \) is measured in meters, and time \( t \) is measured in seconds. Find the velocity of the particle at time \( t = 8 \) (seconds). Write an exact answer with correct units. (8 points)

\[
\begin{align*}
    s(t) &= 3t + \frac{2}{\sqrt[3]{t^2}} = 3t + 2t^{-2/3} \\
    v(t) &= s'(t) = 3 + 2 \left( -\frac{2}{3} t^{-5/3} \right) = 3 - \frac{4}{3} t^{-5/3} = 3 - \frac{4}{3t^{5/3}} \Rightarrow \\
    v(8) &= 3 - \frac{4}{3 \left( \frac{8}{3} \right)^{5/3}} = 3 - \frac{4}{3 \left( \frac{8}{3} \right)^{5/3}} = 3 - \frac{4}{3 \cdot \frac{4}{96}} = 3 - \frac{1}{24}
\end{align*}
\]

\[
\begin{align*}
    v(8) &= \frac{23}{24} \text{ or } \frac{71}{24} \text{ meters per second}
\end{align*}
\]
3) If \( f(x) = \sqrt{9 - x^2} \), is \( f \) differentiable on the interval \([-3, 3]\)? Box in one: (1 point). Yes No

The graph of \( y = f(x) \) has (one-sided) vertical tangent lines at \( x = \pm 3 \):
Also, \( f'(x) \) is undefined at \( x = \pm 3 \):
\[
f(x) = \sqrt{9 - x^2} = (9 - x^2)^{1/2} \Rightarrow f'(x) = \frac{1}{2} (9 - x^2)^{-1/2} (-2x) = -\frac{x}{\sqrt{9 - x^2}}
\]

4) Let \( f(x) = x^2 + x \). These are linearization problems. (11 points total)
a) Find an equation of the tangent line to the graph of \( y = f(x) \) at the point on the graph (in the usual xy-plane) where \( x = 3 \). Use any form. (8 points)

Find the y-coordinate of the point of tangency on the graph.
\[ x_1 = 3 \Rightarrow y_1 = f(3) = (3)^2 + 3 = 12 \]

Find the slope, \( m \), of the tangent line there.
\[ f'(x) = 2x + 1 \Rightarrow m = f'(3) = 2(3) + 1 = 7 \]

Point-slope form of the tangent line:
\[
y - y_1 = m(x - x_1) \Rightarrow y - 12 = 7(x - 3)
\]

Slope-intercept form:
\[
y = mx + b \Rightarrow \quad y = 7x - 9
\]

b) Use your tangent line from a) to give a linear approximation for the value of \( f(2.85) \). Give your answer as a decimal written out to two decimal places. (3 points)

Using Point-slope form:
\[
y = 7(2.85) - 9 = 10.95
\]

Using Slope-intercept form:
\[
y = 10.95
\]

Note: In fact, \( f(2.85) = 10.9725 \). Our approximation is correct to one decimal place.
5) Find the indicated derivatives. (21 points total; 7 points each)

a) Find \( \frac{d}{dr} \left( \sqrt[3]{r^6 - \frac{5}{r^3} + 6} \right) \). Simplify. Do \textbf{not} leave negative exponents in your final answer. Your final answer does \textbf{not} need to be a single fraction.

\[
\frac{d}{dr} \left( \sqrt[3]{r^6 - \frac{5}{r^3} + 6} \right) = D_r \left( r^{6/7} - 5r^{-3} + 6 \right) = \frac{6}{7} r^{\frac{6}{7} - 1} - 5(-3)(-3)^{-1}
\]

\[
= \frac{6}{7} r^{-\frac{1}{7}} + 15r^{-4} = \frac{6}{7} r^{-\frac{1}{7}} + \frac{15}{r^4}, \quad \text{or} \quad \frac{6r^{27/7} + 105}{7r^4}, \quad \text{or} \quad \frac{3(2r^{27/7} + 35)}{7r^4}
\]

b) Let \( f(x) = 3x^7 \sin(x^2) \). Find \( f'(x) \). Simplify. You do \textbf{not} have to factor your final answer.

Apply the Product Rule. Then, use the Generalized Trigonometric Rules for the second term.

\[
f'(x) = \left[ D_x (3x^7) \right] \cdot \left[ \sin(x^2) \right] + \left[ 3x^7 \right] \cdot \left[ D_x (\sin(x^2)) \right]
\]

\[
= \left[ 21x^6 \right] \cdot \sin(x^2) + \left[ 3x^7 \right] \cdot \cos(x^2) \cdot \left[ D_x (x^2) \right]
\]

\[
= 21x^6 \sin(x^2) + 6x^8 \cos(x^2), \quad \text{or} \quad 3x^6 \left[ 7 \sin(x^2) + 2x^2 \cos(x^2) \right]
\]

c) Find \( D_\alpha \left[ \csc^3 \left( 7\alpha + \frac{\pi}{4} \right) \right] \). Simplify.

Rewrite, and then apply the Generalized Power Rule.

\[
D_\alpha \left[ \csc^3 \left( 7\alpha + \frac{\pi}{4} \right) \right] = D_\alpha \left[ \csc \left( 7\alpha + \frac{\pi}{4} \right)^3 \right]
\]

\[
= 3 \left[ \csc \left( 7\alpha + \frac{\pi}{4} \right) \right]^2 \cdot \left[ D_\alpha \left( \csc \left( 7\alpha + \frac{\pi}{4} \right) \right) \right]
\]

\[
= 3 \left[ \csc \left( 7\alpha + \frac{\pi}{4} \right) \right]^2 \cdot \left[ -\csc \left( 7\alpha + \frac{\pi}{4} \right) \cot \left( 7\alpha + \frac{\pi}{4} \right) \right] \cdot \left[ D_\alpha \left( 7\alpha + \frac{\pi}{4} \right) \right]
\]

\[
= -21 \csc^3 \left( 7\alpha + \frac{\pi}{4} \right) \cot \left( 7\alpha + \frac{\pi}{4} \right)
\]
6) Find \( D_x \left( \frac{7x+1}{\sqrt{3x^2 + 5}} \right) \). Simplify. Your final answer must have the form:

\[
\text{an } x \text{ term and a constant term} \quad \left( \frac{3x^2 + 5}{} \right)^{\text{some exponent}}. \quad (12 \text{ points})
\]

\[
D_x \left( \frac{7x+1}{\sqrt{3x^2 + 5}} \right) = \frac{\text{Lo} \cdot D(\text{Hi}) - \text{Hi} \cdot D(\text{Lo})}{(\text{Lo})^2} \quad \text{(Quotient Rule)}
\]

\[
= \frac{\left[ \sqrt{3x^2 + 5} \right] \cdot D_x(7x+1) - [7x+1] \cdot \left[ D_x \left( \frac{7x+1}{\sqrt{3x^2 + 5}} \right) \right]}{\left( \sqrt{3x^2 + 5} \right)^2}
\]

\[
= \frac{\left[ \sqrt{3x^2 + 5} \right] \cdot 7 - [7x+1] \cdot \left[ \frac{1}{2} \left( 3x^2 + 5 \right)^{-1/2} \cdot D_x \left( 3x^2 + 5 \right) \right]}{3x^2 + 5}
\]

\[
= \frac{\left[ \sqrt{3x^2 + 5} \right] \cdot 7 - [7x+1] \cdot \left[ \frac{1}{2} \right] \cdot \left( 3x^2 + 5 \right)^{-1/2} \cdot 6x}{3x^2 + 5}
\]

\[
= \frac{7\left( 3x^2 + 5 \right)^{1/2} - 3x \cdot [7x+1]}{3x^2 + 5} \cdot \left[ \left( 3x^2 + 5 \right)^{-1/2} \right] \cdot [6x]
\]

Let \( u = \left( 3x^2 + 5 \right) \).

\[
= \frac{7u^{1/2} - 3x \cdot [7x+1]}{u} \cdot \left[ u^{-1/2} \right] \cdot [6x]
\]

Method 1: Factor the numerator before dividing powers of \( u \).

\[
= \frac{7u^{1/2} - 3x \cdot [7x+1]}{u} \quad \left( \text{Now, } \frac{u^{-1/2}}{u} = u^{-\frac{1}{2} - 1} = u^{-\frac{3}{2}} = \frac{1}{u^{3/2}} \right)
\]

\[
= \frac{7u^{1/2} - 3x \cdot [7x+1]}{u^{3/2}} \quad \left[ \text{Do not divide powers of } u \text{ here. Sub back } u = \left( 3x^2 + 5 \right) \right]
\]

\[
= \frac{7(3x^2 + 5) - 21x^2 - 3x}{(3x^2 + 5)^{3/2}} \quad \quad \quad \quad = \quad \quad \frac{21x^2 + 35 - 21x^2 - 3x}{(3x^2 + 5)^{3/2}} \quad \quad = \quad \quad \frac{35 - 3x}{(3x^2 + 5)^{3/2}}
\]
Method 2: Rewrite as a compound (or complex) fraction.

\[
7\sqrt{u} - \frac{3x \left[ 7x + 1 \right]}{\sqrt{u}} = \frac{7\sqrt{u} - 3x \left[ 7x + 1 \right]}{\sqrt{u}} = \frac{7u - 3x \left[ 7x + 1 \right]}{u^{3/2}} \cdot \sqrt{u} = \frac{7u - 3x \left[ 7x + 1 \right]}{u^{3/2}} \cdot \sqrt{u} \quad \text{(Distribute } \sqrt{u} \text{ to both terms in red.)}
\]

\[
\frac{7u - 3x \left[ 7x + 1 \right]}{u^{3/2}} \cdot \sqrt{u} = \left( \frac{7u - 3x \left[ 7x + 1 \right]}{u^{3/2}} \right) \cdot \sqrt{u} = \frac{7\left( 3x^2 + 5 \right) - 3x \left[ 7x + 1 \right]}{\left( 3x^2 + 5 \right)^{3/2}} = \frac{21x^2 + 35 - 21x^2 - 3x}{\left( 3x^2 + 5 \right)^{3/2}} = \frac{35 - 3x}{\left( 3x^2 + 5 \right)^{3/2}}
\]

7) Prove that \( D_x \left[ \cot(x) \right] = -\csc^2(x) \) without using the limit definition of the derivative. Among the trigonometric derivatives, you may use only the derivatives of \( \sin(x) \) and \( \cos(x) \) without proof. Show all work! (8 points)

\[
\frac{D_x \left[ \cot(x) \right]}{D_x \left[ \cos(x) \right]} \quad \text{(by the Quotient Identities)}
\]

\[
D_x \left[ \frac{\cos(x)}{\sin(x)} \right] = \frac{\text{Lo} \cdot D(\text{Hi}) - \text{Hi} \cdot D(\text{Lo})}{\text{Hi}^2} \quad \text{(by the Quotient Rule for Differentiation)}
\]

\[
= \frac{\sin(x) \cdot \left( D_x \left[ \cos(x) \right] \right) - \cos(x) \cdot \left( D_x \left[ \sin(x) \right] \right)}{\left[ \sin(x) \right]^2}
\]

\[
= \frac{\left[ \sin(x) \right] \cdot \left[ -\sin(x) \right] - \left[ \cos(x) \right] \cdot \left[ \cos(x) \right]}{\left[ \sin(x) \right]^2}
\]

\[
= \frac{-\sin^2(x) - \cos^2(x)}{\sin^2(x)}
\]

\[
= -\frac{\sin^2(x) + \cos^2(x)}{\sin^2(x)} \quad \text{[Can: } -\left( \frac{\sin^2(x)}{\sin^2(x)} + \frac{\cos^2(x)}{\sin^2(x)} \right) = -\left( 1 + \cot^2(x) \right) = -\csc^2(x) \text{]}
\]

\[
= -\frac{1}{\sin^2(x)} \quad \text{(by the Pythagorean Identities)}
\]

\[
= -\csc^2(x) \quad \text{(by the Reciprocal Identities)}
\]

Q.E.D.
8) Let \( f(x) = (x^2 - 4)^3 \). (10 points total)

a) Find \( f'(x) \). (3 points)

\[
f'(x) = \left[ 3(x^2 - 4)^2 \right] \left[ 2x \right] = 6x(x^2 - 4)^2
\]

(Generalized Power Rule)

b) Find the points on the graph of \( y = f(x) \) (in the usual xy-plane) at which the tangent line is horizontal. (7 points)

Solve \( f'(x) = 0 \) for \( x \), since horizontal lines have slope 0.

\[
6x(x^2 - 4)^2 = 0 \quad \Rightarrow \quad x^2 - 4 = 0
\]

\[
x = 0 \quad \text{or} \quad x^2 = 4 \quad \Rightarrow \quad x = \pm 2
\]

Find the corresponding \( y \)-coordinates by evaluating \( f \) at those \( x \)-values:

\[
f(0) = \left[ (0)^2 - 4 \right]^3 = [-4]^3 = -64
\]

\[
f(2) = \left[ (2)^2 - 4 \right]^3 = [0]^3 = 0
\]

\[
f(-2) = \left[ (-2)^2 - 4 \right]^3 = [0]^3 = 0
\]

The points we want are: \( (0, -64), (2, 0), \) and \( (-2, 0) \).

Note: \( f \) is an even function. Here is the graph of \( y = f(x) \):

9) Consider the given equation \( 6y + 4x^5\ y^2 - \cos(y) = 12 \). Assume that it “determines” an implicit differentiable function \( f \) such that \( y = f(x) \).

Find \( \frac{dy}{dx} \) (you may use the \( y' \) notation, instead). Use implicit differentiation, as in class. (11 points)

\[
6y + 4x^5\ y^2 - \cos(y) = 12 \quad \Rightarrow
\]

\[
D_x \left[ 6y + 4x^5\ y^2 - \cos(y) \right] = D_x(12)
\]

\[
6y' + \left[ D_x(4x^5) \right] \cdot y^2 + \left[ 4x^5 \right] \cdot D_x(y^2) + \left[ \sin(y) \right] \cdot y' = 0
\]

\[
6y' + 20x^4 \cdot y^2 + \left[ 4x^5 \right] \cdot 2yy' + y' \sin(y) = 0
\]

\[
6y' + 20x^4 \cdot y^2 + 8x^5 yy' + y' \sin(y) = 0
\]
Now, isolate the terms with \( y' \) on one side.

\[
6y' + 8x^5y' + y'\sin(y) = -20x^4y^2
\]

Factor out \( y' \) on the left side.

\[
y'^{6 + 8x^5 + \sin(y)} = -20x^4y^2
\]

Divide to solve for \( y' \).

\[
\frac{dy}{dx} \quad \text{or} \quad y' = \frac{-20x^4y^2}{6 + 8x^5 + \sin(y)}
\]

10) Air is being pumped into a spherical balloon at the rate of 2.3 cubic feet per minute. The balloon maintains a spherical shape throughout. At what rate is the radius of the balloon changing when the radius is 30 inches in length?

- In your final answer, give the appropriate units, and round off your answer as a decimal to four significant digits. (Round off intermediate results to at least four significant digits.)

- Hint 1: If you forgot the “key formula” here, you can buy it from me for 2 points. You can’t get negative points for this problem.

- Hint 2: One foot is equivalent to 12 inches. (11 points)

**Step 1: Read the problem!**

**Step 2: Define variables / General diagram (optional here)**

Let \( t = \) time in minutes.

Let \( V = \) the volume of the balloon [at time \( t \)].

Let \( r = \) the radius of the balloon [at time \( t \)].

**Step 3: Given / Find (what?) at the instant of interest.**

Given: \( \frac{dV}{dt} = 2.3 \left( \frac{\text{ft}^3}{\text{min}} \right) \).

Find: \( \frac{dr}{dt} \) when \( r = 30 \text{ in.} \)

Convert units: \( 30 \text{ in} \left( \frac{1 \text{ ft}}{12 \text{ in}} \right) = 2.5 \text{ ft.} \)

**Step 4: Key formula.** Volume \( V = \frac{4}{3} \pi r^3 \).

**Step 5: Perform Implicit Differentiation on the formula in Step 4.**

\[
V = \frac{4}{3} \pi r^3 \quad \Rightarrow
\]

\[
D_t(V) = D_t \left( \frac{4}{3} \pi r^3 \right)
\]

\[
\frac{dV}{dt} = \frac{4}{3} \pi \left( 3r^2 \cdot \frac{dr}{dt} \right) \quad \text{(by the Chain Rule)}
\]

\[
\frac{dV}{dt} = 4\pi r^2 \cdot \frac{dr}{dt}
\]
Step 6: Plug in values at the instant of interest.

Step 7: Solve for \( \frac{dr}{dt} \).

\[
2.3 = 4\pi (2.5)^2 \cdot \frac{dr}{dt}
\]
\[
2.3 = 25\pi \cdot \frac{dr}{dt}
\]
\[
\frac{dr}{dt} = \frac{2.3}{25\pi} = \frac{23}{250\pi} \quad \text{(Type: } 23 \div 250 \div \pi = \text{)}
\]
\[
\frac{dr}{dt} \approx 0.02928 \text{ ft/min}
\]

Step 8: Conclusion.

The radius of the balloon is growing at the rate of about \( 0.02928 \text{ ft/min} \) when its radius is 30 inches.

Note on Unit Conversions:

Given: \( \frac{dV}{dt} = \left( \frac{2.3 \text{ ft}^3}{\text{min}} \right) \left( \frac{12 \text{ in}}{1 \text{ ft}} \right)^3 = \left( \frac{2.3 \text{ ft}^3}{\text{min}} \right) \left( \frac{1728 \text{ in}^3}{1 \text{ ft}^3} \right) = 3974.4 \text{ in}^3/\text{min} \)

Find: \( \frac{dr}{dt} \) when \( r = 30 \text{ in} \).

Step 7:

\[
3974.4 = 4\pi (30)^2 \cdot \frac{dr}{dt}
\]
\[
3974.4 = 3600\pi \cdot \frac{dr}{dt}
\]
\[
\frac{dr}{dt} = \frac{3974.4}{3600\pi} \approx \frac{39744}{125\pi} = \frac{138}{250 \approx} \frac{0.3514}{\text{min}}
\]

Observe:

\[
\left( \frac{0.3514 \text{ in}}{\text{min}} \right) \left( \frac{1 \text{ ft}}{12 \text{ in}} \right) = 0.02928 \text{ ft/min}, \text{ our original answer.}
\]