

QUIZ ON CHAPTER 3 - SOLUTIONS

DERIVATIVES; MATH 150 – SPRING 2017 – KUNIYUKI

105 POINTS TOTAL, BUT 100 POINTS = 100%

- 1) Use the limit definition of the derivative to prove that $D_x[\cos(x)] = -\sin(x)$, as in class. Show all steps! (12 points)

$$\begin{aligned}
 &\text{Let } f(x) = \cos(x). \\
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cos(x+h) - \cos(x)}{h} \\
 &\quad \text{from Sum Identity for cosine} \\
 &= \lim_{h \rightarrow 0} \frac{\cos(x)\cos(h) - \sin(x)\sin(h) - \cos(x)}{h} \\
 &\quad \text{Group terms with } \cos(x). \\
 &= \lim_{h \rightarrow 0} \frac{[\cos(x)\cos(h) - \cos(x)] - \sin(x)\sin(h)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{[\cos(x)][\cos(h)-1] - [\sin(x)][\sin(h)]}{h} \\
 &= \lim_{h \rightarrow 0} \left([\cos(x)] \underbrace{\left[\frac{\cos(h)-1}{h} \right]}_{\rightarrow 0} - [\sin(x)] \underbrace{\left[\frac{\sin(h)}{h} \right]}_{\rightarrow 1} \right) \quad (\text{Form fractions in } h.) \\
 &= -\sin(x) \quad (\text{Q.E.D.})
 \end{aligned}$$

- 2) A particle moving along a coordinate line has as its position function s , where $s(t) = \sin(4t)$. Position $s(t)$ is measured in miles, and time t is measured in hours. Write exact answers with correct units. (10 points total)

- a) Find the velocity of the particle at time $t = \frac{\pi}{24}$.

$$\begin{aligned}
 v(t) &= s'(t) = D_t[\sin(4t)] = [\cos(4t)][4] = 4\cos(4t) \Rightarrow \\
 v\left(\frac{\pi}{24}\right) &= 4\cos\left[4\left(\frac{\pi}{24}\right)\right] = 4\cos\left[\frac{\pi}{6}\right] = 4\left(\frac{\sqrt{3}}{2}\right) = \boxed{2\sqrt{3} \frac{\text{mi}}{\text{hr}} \text{ (or mph)}}
 \end{aligned}$$

- b) Find the acceleration of the particle at time $t = \frac{\pi}{24}$.

$$\begin{aligned}
 a(t) &= v'(t) = D_t[4\cos(4t)] = [-4\sin(4t)][4] = -16\sin(4t) \Rightarrow \\
 a\left(\frac{\pi}{24}\right) &= -16\sin\left[4\left(\frac{\pi}{24}\right)\right] = -16\sin\left[\frac{\pi}{6}\right] = -16\left(\frac{1}{2}\right) = \boxed{-8 \frac{\text{mi}}{\text{hr}^2}}
 \end{aligned}$$

3) (2 points). Which of the following equations has a cusp on its graph at $(0, 0)$?

Box in one:

$$y = x^{3/5}$$

$$y = x^{4/5}$$

$$y = |x|$$

See Ex.14 in the Notes for Section 3.2. The graph of $y = x^{4/5}$ is similar to that of $y = x^{2/3}$.

4) Let $f(x) = (3x - 1)^3$. Consider the graph of $y = f(x)$ in the usual xy -plane.

Also consider the point P on that graph with $x = 1$. These are linearization problems. (11 points total)

a) Find an equation of the tangent line to the graph at the point P . You may use any appropriate form. (8 points)

Find the y -coordinate of P , the point of tangency. Find y_1 where $x_1 = 1$.

$$f(1) = [3(1) - 1]^3 = [2]^3 = 8 \Rightarrow y_1 = 8$$

Find the slope, m , of the tangent line at P .

$$f'(x) = 3(3x - 1)^2 \cdot [D_x(3x - 1)] = 3(3x - 1)^2(3) = 9(3x - 1)^2 \Rightarrow$$

$$f'(1) = 9[3(1) - 1]^2 = 9(2)^2 = 36 \Rightarrow m = 36$$

Point-slope form of the tangent line:

$$y - y_1 = m(x - x_1)$$

$$y - 8 = 36(x - 1)$$

Slope-intercept form:

$$y - 8 = 36x - 36$$

$$y = 36x - 28$$

$$\begin{aligned} \text{or } y &= mx + b \Rightarrow \\ (8) &= (36)(1) + b \Rightarrow \\ b &= -28 \Rightarrow \\ y &= 36x - 28 \end{aligned}$$

b) Use your tangent line from a) to give a linear approximation for the value of $f(0.93)$. Give your answer as a decimal written out to two decimal places. (3 points)

Using Point-slope form:

$$y - 8 = 36(0.93 - 1)$$

$$y - 8 = 36(-0.07)$$

$$y - 8 = -2.52$$

$$y = 8 - 2.52$$

$$y = 5.48$$

Using Slope-intercept form:

$$y = 36(0.93) - 28 = 5.48$$

Note: In fact, $f(0.93) = 5.735339$.

5) Find the indicated derivatives. (27 points total)

- a) $D_x \left[\frac{\cot(x) + 3}{x^2 - 1} \right]$. Use the Quotient Rule! Your final answer does not have to be in simplified form, aside from arithmetic. (8 points)

$$\begin{aligned} D_x \left[\frac{\cot(x) + 3}{x^2 - 1} \right] &= \frac{\text{Lo} \cdot \mathbf{D(Hi)} - \text{Hi} \cdot \mathbf{D(Lo)}}{(\text{Lo})^2} \quad (\text{by Quotient Rule for Differentiation}) \\ &= \frac{[x^2 - 1] \cdot [D_x(\cot(x) + 3)] - [\cot(x) + 3] \cdot [D_x(x^2 - 1)]}{(x^2 - 1)^2} \\ &= \frac{[x^2 - 1] \cdot [-\csc^2(x)] - [\cot(x) + 3] \cdot [2x]}{(x^2 - 1)^2}, \text{ or } \frac{-x^2 \csc^2(x) + \csc^2(x) - 2x \cot(x) - 6x}{(x^2 - 1)^2} \end{aligned}$$

- b) $\frac{d}{dw} \left[(5w^3 - w)^{-7} \right]$. Do not leave negative exponents in your final answer. Simplify arithmetic. (5 points)

$$\begin{aligned} \frac{d}{dw} \left[(5w^3 - w)^{-7} \right] &= [-7(5w^3 - w)^{-8}] \cdot [D_w(5w^3 - w)] \quad (\text{by Generalized Power Rule}) \\ &= -7(5w^3 - w)^{-8} (15w^2 - 1) = \frac{7(1 - 15w^2)}{(5w^3 - w)^8}, \text{ or } \frac{7(1 - 15w^2)}{w^8(5w^2 - 1)^8} \end{aligned}$$

- c) Let $f(x) = (x + 2)^5 \cos(5x^2)$. Find $f'(x)$. Simplify. You do not have to factor your final answer. (8 points)

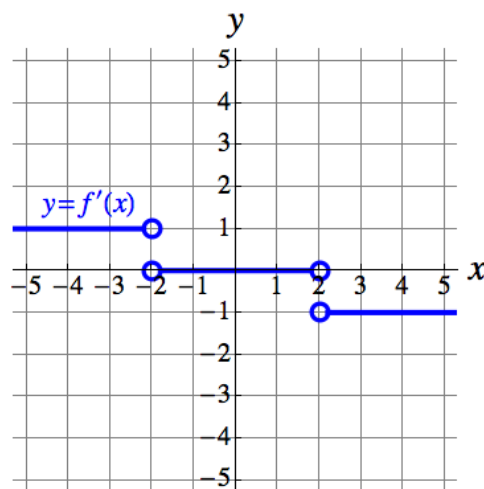
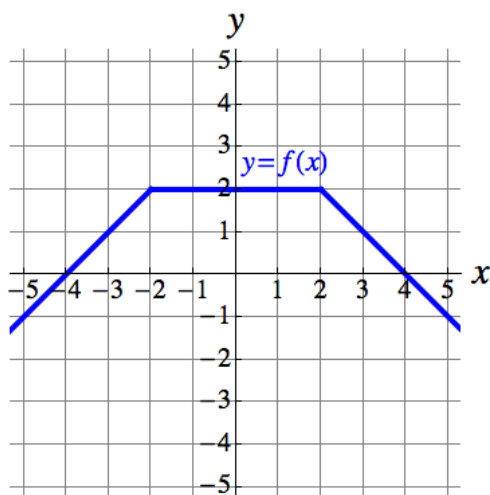
Use the Product Rule. Then, use the Generalized Power and Trigonometric Rules.

$$\begin{aligned} f'(x) &= \left(D_x \left[(x + 2)^5 \right] \right) \cdot [\cos(5x^2)] + [(x + 2)^5] \cdot \left(D_x [\cos(5x^2)] \right) \\ &= [5(x + 2)^4 \cdot D_x(x + 2)] \cdot [\cos(5x^2)] + [(x + 2)^5] \cdot [-\sin(5x^2)] \cdot [D_x(5x^2)] \\ &= [5(x + 2)^4 (1)] \cdot [\cos(5x^2)] + [(x + 2)^5] \cdot [-\sin(5x^2)] \cdot [10x] \\ &= [5(x + 2)^4 \cos(5x^2) - 10x(x + 2)^5 \sin(5x^2)], \text{ or } 5(x + 2)^4 [\cos(5x^2) - 2x(x + 2) \sin(5x^2)] \end{aligned}$$

- d) Find $D_\theta [\sec^9(3\theta)]$. Simplify. (6 points)

$$\begin{aligned} D_\theta [\sec^9(3\theta)] &= D_\theta \left([\sec(3\theta)]^9 \right) = 9[\sec(3\theta)]^8 \cdot D_\theta [\sec(3\theta)] \quad (\text{Gen. Power Rule}) \\ &= 9[\sec(3\theta)]^8 \cdot [\sec(3\theta) \tan(3\theta)] \cdot \underbrace{[D_\theta(3\theta)]}_{=[3]} \quad (\text{Gen. Trig Rule}) = \boxed{27 \sec^9(3\theta) \tan(3\theta)} \end{aligned}$$

- 6) The graph of $y = f(x)$ is given below, to the left. Sketch the graph of $y = f'(x)$ in the grid to the right. (6 points)



$$f(x) = \begin{cases} x+4, & x < -2 \quad (\text{Think: run 1} \Rightarrow \text{rise 1}) \\ 2, & -2 \leq x \leq 2 \quad (\text{Think: constant piece}) \\ 4-x, & x > 2 \quad (\text{Think: run 1} \Rightarrow \text{drop 1}) \end{cases}$$

$$f'(x) = \begin{cases} 1, & x < -2 \\ 0, & -2 < x < 2 \\ -1, & x > 2 \end{cases}$$

Graph #1 has corners at $x = -2$ and $x = 2$, so f is not differentiable (f' is undefined) there.

- 7) Prove that $D_x[\csc(x)] = -\csc(x)\cot(x)$ without using the limit definition of the derivative. Among the trigonometric derivatives, you may use only the derivatives of $\sin(x)$ and $\cos(x)$ without proof. Show all work! (8 points)

$$\begin{aligned} D_x[\csc(x)] &= D_x\left[\frac{1}{\sin(x)}\right] \quad (\text{by a Reciprocal Identity from Trigonometry}) \\ &= \frac{\text{Lo} \cdot \mathbf{D(Hi)} - \text{Hi} \cdot \mathbf{D(Lo)}}{(\text{Lo})^2} \quad (\text{Quotient Rule}) \quad \text{or} \quad -\frac{\mathbf{D(Lo)}}{(\text{Lo})^2} \quad (\text{Reciprocal Rule}) \\ &= \frac{[\sin(x)] \cdot [\mathbf{D_x(1)}] - [1] \cdot (\mathbf{D_x[\sin(x)]})}{[\sin(x)]^2} \quad \text{or} \quad -\frac{\mathbf{D_x[\sin(x)]}}{[\sin(x)]^2} \\ &= \frac{\overbrace{[\sin(x)] \cdot [\mathbf{0}]}^0 - [1] \cdot [\mathbf{\cos(x)}]}{[\sin(x)]^2} \quad \text{or} \quad -\frac{\mathbf{\cos(x)}}{[\sin(x)]^2} \\ &= -\frac{\cos(x)}{\sin^2(x)} \\ &= -\frac{1}{\sin(x)} \cdot \frac{\cos(x)}{\sin(x)} \quad (\text{by "Peeling" / Factoring}) \\ &= -\csc(x)\cot(x) \quad (\text{by Reciprocal and Quotient Identities from Trigonometry}) \end{aligned}$$

Q.E.D.

- 8) Consider the given equation $3x^5 - 4y + x \cos(y) = 7$. Assume that it “determines” an implicit differentiable function f such that $y = f(x)$. Find $\frac{dy}{dx}$ (you may use the y' notation, instead). Use implicit differentiation, as in class. (11 points)

$$\begin{aligned}
 3x^5 - 4y + x \cos(y) &= 7 \quad \Rightarrow \\
 D_x \left(3x^5 - 4y + \underbrace{x \cos(y)}_{\text{Use the Product Rule!}} \right) &= D_x(7) \quad \Rightarrow \\
 15x^4 - 4y' + [D_x(x)] \cdot [\cos(y)] + [x] [D_x(\cos(y))] &= 0 \\
 15x^4 - 4y' + [1] \cdot [\cos(y)] + [x] [-\sin(y)] [y'] &= 0 \\
 15x^4 - 4y' + \cos(y) - xy' \sin(y) &= 0
 \end{aligned}$$

Now, isolate the terms with y' on one side.
(We will also multiply both sides by -1 .)

$$\begin{aligned}
 -4y' - xy' \sin(y) &= -15x^4 - \cos(y) \\
 4y' + xy' \sin(y) &= 15x^4 + \cos(y)
 \end{aligned}$$

Factor out y' on the left side.

$$y' [4 + x \sin(y)] = 15x^4 + \cos(y)$$

Divide to solve for y' .

$$\boxed{\frac{dy}{dx} \text{ or } y' = \frac{15x^4 + \cos(y)}{4 + x \sin(y)}}$$

- 9) $A = \pi r^2$, where r is a differentiable function of t . Find $\frac{dA}{dt}$ when $r = 3$ and

$$\frac{dr}{dt} = 5. \text{ (6 points)}$$

$$\begin{aligned}
 A &= \pi r^2 \quad \Rightarrow \\
 D_t(A) &= D_t(\pi r^2) \\
 \frac{dA}{dt} &= \pi \cdot 2r \left(\frac{dr}{dt} \right) \\
 \frac{dA}{dt} &= 2\pi r \left(\frac{dr}{dt} \right) \quad \Rightarrow
 \end{aligned}$$

At the instant of interest,

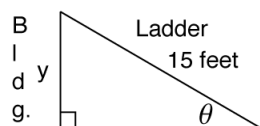
$$\frac{dA}{dt} = 2\pi(3)(5) = \boxed{30\pi}$$

- 10) A 15-foot-long ladder is leaning at an angle against a tall vertical building that is standing upright and perpendicular from the flat ground. The bottom of the ladder is sliding away from the building in such a way that the top of the ladder is falling at a rate of 0.27 feet per minute down the building. Find the rate at which the angle (the “angle of elevation”) between the ground and the ladder is changing when the angle is 37° .

• Write your final answer in degrees per minute, and round off your answer as a decimal to four significant digits. (Round off intermediate results to at least four significant digits.) (12 points)

Step 1: Read the problem!

Step 2: Define variables / General diagram.



Step 3: Given / Find (what?) at the instant of interest.

$$\text{Given: } \frac{dy}{dt} = -0.27 \frac{\text{ft}}{\text{min}}$$

$$\text{Find: } \frac{d\theta}{dt} \text{ when } \theta = 37^\circ.$$

Step 4: Key formula.

$$\sin(\theta) = \frac{y}{15}$$

$$y = 15 \sin(\theta) \Rightarrow$$

Step 5: Perform Implicit Differentiation on the formula in Step 4.

$$D_t(y) = D_t[15 \sin(\theta)]$$

$$\frac{dy}{dt} = [15 \cos(\theta)] \cdot \frac{d\theta}{dt}$$

Step 6: Plug in values at the instant of interest.

$$\frac{dy}{dt} = [15 \cos(\theta)] \cdot \frac{d\theta}{dt} \Rightarrow$$

$$(-0.27) = [15 \cos(37^\circ)] \cdot \frac{d\theta}{dt}$$

Step 7: Solve for $\frac{d\theta}{dt}$.

$$\frac{d\theta}{dt} = \frac{-0.27}{15 \cos(37^\circ)}$$

$$\approx -0.0225384 \frac{\text{radians}}{\text{min}}$$

$$= \left(-0.0225384 \frac{\text{radians}}{\text{min}} \right) \cdot \left(\frac{180^\circ}{\pi \text{ radians}} \right)$$

$$\approx \boxed{-1.291 \frac{\text{degrees}}{\text{min}}}$$

Step 8: Conclusion.

The angle is decreasing at a rate of about 1.291 degrees per minute when the angle is 37° .