QUIZ ON CHAPTER 4 - SOLUTIONS

APPLICATIONS OF DERIVATIVES; MATH 150 – FALL 2016 – KUNIYUKI
105 POINTS TOTAL, BUT 100 POINTS = 100%

1) Consider $f(x)=\dfrac{x+5}{16-x^2}$ and the graph of $y=f(x)$ in the usual xy-plane in parts a) through f). If the answer to a part is none, write "NONE." (25 points)

a) Find <u>and box in</u> all critical numbers of f. Show all work. (9 points)

$\text{Dom}(f)=\left\{x\in\mathbb{R}\,\middle|\,x\neq\pm 4\right\}$, since those are the real values for which $16-x^2\neq 0$.

$$f'(x)=\frac{\text{Lo}\cdot\text{D}(\text{Hi})-\text{Hi}\cdot\text{D}(\text{Lo})}{(\text{Lo})^2}=\frac{\left[16-x^2\right]\cdot\left[D_x(x+5)\right]-\left[x+5\right]\cdot\left[D_x\left(16-x^2\right)\right]}{\left(16-x^2\right)^2}$$

$$=\frac{\left[16-x^2\right]\cdot[1]-[x+5]\cdot[-2x]}{\left(16-x^2\right)^2}=\frac{\left(16-x^2\right)-\left(-2x^2-10x\right)}{\left(16-x^2\right)^2}=\frac{16-x^2+2x^2+10x}{\left(16-x^2\right)^2}$$

$$=\frac{x^2+10x+16}{\left(16-x^2\right)^2}=\frac{(x+8)(x+2)}{\left(16-x^2\right)^2}$$

The real values of x that make f' DNE (namely, ± 4) are **not** in $\text{Dom}(f)$, so they **cannot** be critical numbers (CNs) of f.

Now, solve $f'(x)=0$ for real x in $\text{Dom}(f)$.

$$f'(x)=0$$
$$\frac{(x+8)(x+2)}{\left(16-x^2\right)^2}=0$$
$$(x+8)(x+2)=0,\quad\left(16-x^2\neq 0;\text{ that is, }x\neq\pm 4\right)$$
$$x=-8\quad\text{or}\quad x=-2$$

Our two real solutions are in $\text{Dom}(f)$, so they are critical numbers (CNs) of f.

The critical numbers (CNs) of f are: $\boxed{-8\ \text{ and }\ -2}$.

b) Find the x-intercept(s) on the graph of $y=f(x)$. (2 points)

Find the real zeros of f. (We set $y=0$, or $f(x)=0$, and solve for x.)

$$\frac{x+5}{16-x^2}=0$$
$$x+5=0\quad\left(\text{and }16-x^2\neq 0\right)$$
$$x=-5$$

The x-intercept is at $\boxed{(-5,0)}$.

c) Find the y-intercept on the graph of $y = f(x)$. (2 points)

(We set $x = 0$ and solve for y. We evaluate:)

$$f(0) = \frac{(0)+5}{16-(0)^2} = \frac{5}{16}\text{, so the } y\text{-intercept is at } \boxed{\left(0, \frac{5}{16}\right)}.$$

d) Find the equation(s) of any horizontal asymptote(s) for the graph of $y = f(x)$. (3 points)

$\frac{x+5}{16-x^2}$ is rational and "bottom-heavy"; the degree of the denominator (2) is greater than that of the numerator (1). The horizontal asymptote is at $\boxed{y = 0}$, the x-axis.

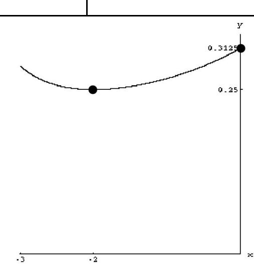
e) Find the equation(s) of any vertical asymptote(s) for the graph of $y = f(x)$. (4 points)

The denominator, $16 - x^2$, has only 4 and -4 as its real zeros. $\frac{x+5}{16-x^2}$ is simplified; the numerator and denominator have no common real zeros and no nontrivial factors in common. The vertical asymptotes are at $\boxed{x = 4 \text{ and } x = -4}$.

f) Find the absolute maximum point (x, y) and the absolute minimum point (x, y) on the graph of $y = f(x)$, if x is restricted to the interval $[-3, 0]$. Indicate which point is the absolute maximum point and which is the absolute minimum point. (5 points)

f is continuous on $\mathbb{R}$ except at -4 and 4. In particular, f is continuous on the closed, bounded interval $[-3, 0]$, so the Extreme Value Theorem (EVT) applies, and there exist absolute maxima and minima on $[-3, 0]$. Our candidates for x are -2 [the critical number (CN) of f in $(-3, 0)$] and the interval endpoints, -3 and 0.

x	$f(x)$	Answers
$a = -3$	$f(-3) = \frac{2}{7} \approx 0.286$	
-2	$f(-2) = \frac{1}{4} = 0.25$	$(-2, 0.25)$ is the A.Min.Pt. on $[-3, 0]$.
$b = 0$	$f(0) = \frac{5}{16} = 0.3125$	$(0, 0.3125)$ is the A.Max.Pt. on $[-3, 0]$.



2) State Rolle's Theorem, including the hypotheses and the conclusion. Write the conclusion using the algebraic notation we used in class; don't just refer to tangent lines or secant lines. (8 points)

> If a function f is continuous on a closed, bounded interval $[a,b]$ and is differentiable on the open interval (a,b), and if $f(a)=f(b)$, then $\exists c\in(a,b)$ such that $f'(c)=0$ (that is, the tangent line is horizontal there). (We assume $a<b$.)

3) Sketch the graph of $y=f(x)$, where $f(x)=x^3-3x^2-24x+13$, in the usual *xy*-plane. (27 points)

- Find all critical numbers of f and label them CNs.
- Find all points at critical numbers.
  Indicate these points on your graph.
- Find all inflection points (if any) and label them IPs.
  Indicate these points on your graph.
- Classify all points at critical numbers as local maximum points,
  local minimum points, or neither.
- Find the *y*-intercept. You do not have to find *x*-intercepts.
- Have your graph show where f is increasing / decreasing and where the
  graph is concave up / concave down. Justify with work, as in class.
- You may round off any non-integers to five significant digits.
- Show all steps, as we have done in class.

Step 1: f is a polynomial function. Therefore,

- $\text{Dom}(f)=\mathbb{R}$.
- f is continuous on $\mathbb{R}$.
- The graph has no vertical asymptotes (VAs) and no holes. Furthermore, since f is nonconstant and nonlinear, the graph has no horizontal asymptotes (HAs) and no slant asymptotes (SAs).

For this f:

- The *y*-intercept is at the point $(0,13)$. This is because $f(0)=13$, the constant term from the given polynomial.
- f is neither even nor odd.

Step 2: Find $f'(x)$ and critical numbers (CNs) of f.

$$f'(x) = 3x^2-6x-24 = 3\left(x^2-2x-8\right) = 3(x+2)(x-4)$$

f' is continuous on $\mathbb{R}$. In particular, it is never undefined ("DNE"). The real solutions of $f'(x)=0$ are given by $x=-2$ and $x=4$, which are in $\text{Dom}(f)$.

The CNs are -2 and 4.

Step 3: Do a sign diagram for f' and classify the points at the CNs.

f is continuous on $\mathbb{R}$, so the First Derivative Test (1st DT) should apply wherever we have CNs. Both f and f' are continuous on $\mathbb{R}$, so we use just the CNs as "fenceposts" on the real number line where f' could change sign.

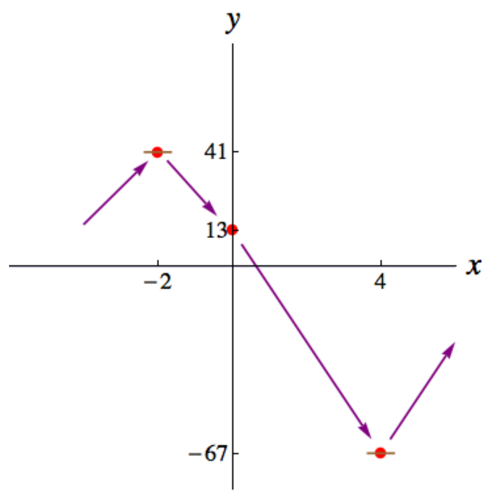
x	Test $x=-3$	$\mathbf{-2}$	Test $x=0$	$\mathbf{4}$	Test $x=5$
f' sign (see below)	$+$	0	$-$	0	$+$
f	$\nearrow$		$\searrow$		$\nearrow$
Classify Points at CNs (Use 1st DT)		**L.Max. Pt.**		**L.Min. Pt.**	
Sub into $f(x)$ to get y		$(-2, f(-2))$ $\mathbf{(-2, 41)}$		$(4, f(4))$ $\mathbf{(4, -67)}$	

$$
\begin{aligned}
f'(x) &= (3)(x+2)(x-4) \quad \Rightarrow \\
f'(-3) &= (+)\ (-)\ \ (-) \ = \ + \\
f'(0) &= (+)\ (+)\ \ (-) \ = \ - \\
f'(5) &= (+)\ (+)\ \ (+) \ = \ +
\end{aligned}
$$

The multiplicities of the zeros of f' are both odd (1), so signs alternate in our "windows." Furthermore, the graph of $y = f'(x)$ is an upward-opening parabola.

Step 4: Sketch a "skeleton graph" for $y = f(x)$ (optional).

Horizontal tangent lines are indicated in brown.



Step 5: Find $f''(x)$ and possible inflection numbers (PINs).

$$
\begin{aligned}
f'(x) &= 3x^2 - 6x - 24 \quad \Rightarrow \\
f''(x) &= 6x - 6 = 6(x-1)
\end{aligned}
$$

f'' is continuous on $\mathbb{R}$. In particular, it is never undefined ("DNE").

The real solution of $f''(x) = 0$ is given by $x = 1$, which is in $\text{Dom}(f)$.

The PIN is 1.

Step 6: Do a sign diagram for f'' and find inflection points (IPs).

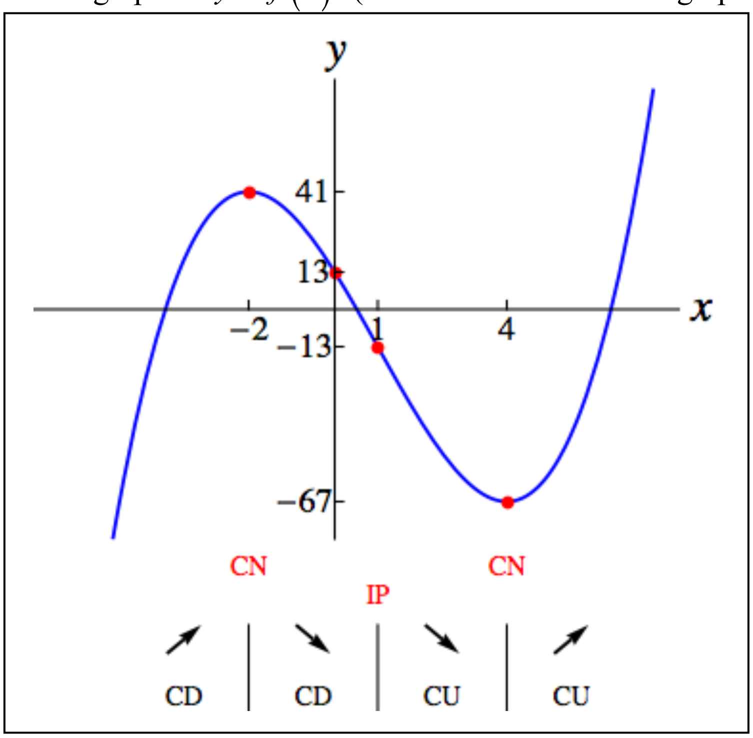
f, f', and f'' are continuous on $\mathbb{R}$, so we use just the PINs as "fenceposts" on the real number line where f'' could change sign.

x	Test $x = 0$	**1**	Test $x = 2$
f'' sign (see below)	**−**		$+$
f graph	CD $(\cap)$		CU $(\cup)$
Inflection Points (IPs)?		**Yes, IP** (f cont. here, concavity changes)	
Sub into $f(x)$ to get y		$(1, f(1))$ $\mathbf{(1, -13)}$	

$$f''(x) = (6)(x-1) \quad \Rightarrow$$
$$f''(0) = (+)\ (-) \quad = \quad -$$
$$f''(2) = (+)\ (+) \quad = \quad +$$

The multiplicity of $x = 1$ as a zero of f'' is 1 (odd), so f'' changes signs. Also, the graph of $y = f''(x)$ is a rising line with x-intercept at $(1, 0)$.

Step 7: Sketch the graph of $y = f(x)$. (Remember our skeleton graph!)



"Zooming out (in the long run)," the graph will resemble a "rising snake." The leading term of $x^3 - 3x^2 - 24x + 13$ is x^3, which has odd degree and a positive leading coefficient.

4) You do <u>not</u> have to show work for these problems. (9 points total)

a) The following is true of the polynomial function f: $f(3)=4,\ f'(3)=0,\ f''(3)=5$. True or False: The point $(3,4)$ must be a local maximum point for the graph of $y=f(x)$. Box in one:

True $\boxed{\text{False}}$

The point $(3,4)$ is a local <u>minimum</u> point by the Second Derivative Test. $f'(3)=0$. The graph of $y=f(x)$ is concave up "at" [actually, "on a neighborhood of"] $x=3$.

b) The following is true of the polynomial function g: $g(6)=-2,\ g'(6)=3,\ g''(6)=-2$. True or False: The point $(6,-2)$ must be a local maximum point for the graph of $y=g(x)$. Box in one:

True $\boxed{\text{False}}$

6 is **not** a critical number (CN) of g, since $g'(6)$ is neither 0 nor undefined ("DNE"). Local maxima must be at critical numbers.

c) The function h has the interval $[6,\ 10]$ as its domain, and h is continuous on that interval. True or False: On the interval $[6,\ 10]$, there is an absolute minimum point for the graph of $y=h(x)$ in the usual *xy*-plane. Box in one:

$\boxed{\text{True}}$ False

This is true by the Extreme Value Theorem (EVT), because h is continuous on the closed, bounded interval $[6,10]$.

5) Let $s(t)$ be the height in feet (at time t in seconds) of a particle that is moving along a vertical line. If $s'(4)=3\dfrac{\text{ft}}{\text{sec}}$, and $s''(4)=-5\dfrac{\text{ft}}{\text{sec}^2}$, what is the particle doing "at" (really, on a neighborhood of) $t=4$ seconds? Box in one: (3 points)

a) The particle is rising and is speeding up.

$\boxed{\text{b) The particle is rising and is slowing down.}}$

c) The particle is falling and is speeding up.

d) The particle is falling and is slowing down.

$s'(4)=v(4)>0$, so the particle's height is increasing, and the particle is rising.

Because $s'(4)=v(4)>0$ and $s''(4)=a(4)<0$, the height is increasing at a decreasing rate, and the particle is slowing down. (v is velocity; a is acceleration.)

6) We want to approximate a root (or zero) of $2x^4 + x - 1$ using Newton's Method with $x_1 = -2$ as our "seed" (our first approximation). (12 points total)

   a) Find x_2, which is our second approximation using Newton's Method. When rounding, use five significant digits; round off your answer to four decimal places.

      Let $f(x) = 2x^4 + x - 1$. Then, $f'(x) = 8x^3 + 1$.

      $x_1 = -2 \quad \Rightarrow$

      $$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = -2 - \frac{f(-2)}{f'(-2)} = -2 - \frac{2(-2)^4 + (-2) - 1}{8(-2)^3 + 1}$$

      $$= -2 - \frac{29}{-63} = -\frac{97}{63} \approx \boxed{-1.5397}$$

   b) Find x_3, which is our third approximation using Newton's Method. When rounding, use five significant digits; round off your answer to four decimal places.

      $$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = -1.5397 - \frac{f(-1.5397)}{f'(-1.5397)}$$

      $$= -1.5397 - \frac{2(-1.5397)^4 + (-1.5397) - 1}{8(-1.5397)^3 + 1} \approx \boxed{-1.2312}$$

      Note 1: The real roots (zeros) of f are -1 and about 0.647799. The two other complex roots are imaginary.

      Note 2: $x_4 \approx -1.0615$, $x_5 \approx -1.0057$, $x_6 \approx -1.0001$, $x_7 \approx -1.0000$, $x_8 \approx -1.0000$. The last two iterates are equal to four decimal places, so we would take -1.0000 as our final approximation.

      Note 3: We could have simplified the iteration formula (for $n \in \mathbb{Z}^+$):

      $$x_{n+1} = x_n - \frac{2x_n^4 + x_n - 1}{8x_n^3 + 1} = \frac{x_n(8x_n^3 + 1)}{8x_n^3 + 1} - \frac{2x_n^4 + x_n - 1}{8x_n^3 + 1} = \frac{6x_n^4 + 1}{8x_n^3 + 1}$$

7) Prove that, among all rectangles with fixed perimeter p, where $p > 0$, the largest in area is a square. (Keep p in your work; don't just pick a numerical value for p.)

   • Show all work, and verify that a square is, in fact, the rectangle with the **absolute** maximum area, as in class.

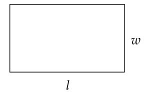
   • Do not use any shortcut precalculus formulas or methods when methods from calculus can be used instead.

   (21 points)

   Step 1: Read the problem.

Step 2: Set up a diagram, a table, etc.

Let l and w be the dimensions of a rectangle with perimeter p.



Step 3: Write the primary equation.

Maximize A, the area of the rectangle.

$$A = l\,w$$

Step 4: Write any relevant secondary (constraint) equations.

Perimeter $p = 2l + 2w$

Step 5: Express A in terms of <u>only one</u> variable, say l.

Solve the constraint equation $p = 2l + 2w$ for w in terms of l, say:

$$2l + 2w = p$$

$$2w = p - 2l$$

$$w = \frac{p}{2} - l \quad \text{(Revised Secondary / Constraint equation)}$$

Express A in terms of l. Incorporate the constraint into the primary equation:

$$A = l\,w$$

$$A = l\left(\frac{p}{2} - l\right)$$

$$A = \frac{p}{2}\,l - l^2$$

Let $f(l) = \frac{p}{2}\,l - l^2$. f is our objective function.

Step 6: Write the "feasible domain" of f.

We require $l \geq 0$. We also require $w \geq 0$, but we will rewrite this in terms of l:

$$w \geq 0$$

$$\frac{p}{2} - l \geq 0 \quad \text{(From the Revised Constraint Equation)}$$

$$\frac{p}{2} \geq l$$

$$l \leq \frac{p}{2}$$

$$\text{Dom}(f) = \left[0, \frac{p}{2}\right].$$

<u>Note</u>: $l = 0$ and $w = 0$ may not make practical sense, but they are permissible, and it is convenient to have a closed interval as our feasible domain.

<u>**Maximize A; that is, Maximize f**</u>

Find critical numbers (CNs) of f in $\left(0, \frac{p}{2}\right)$. Remember that p is a constant!

$$f(l) = \frac{p}{2}\,l - l^2 \quad \Rightarrow$$

$$f'(l) = \frac{p}{2} - 2l \quad \left[\text{never undefined ("DNE") on } \left(0, \frac{p}{2}\right)\right]$$

Solve $f'(l)=0$ for l in $\left(0,\frac{p}{2}\right)$:

$$\frac{p}{2}-2l=0$$
$$\frac{p}{2}=2l$$
$$2l=\frac{p}{2}$$
$$l=\frac{p}{4}$$

Now, $\frac{p}{4}\in\left(0,\frac{p}{2}\right)$, so it is a critical number (CN).

If $l=\frac{p}{4}$, then, by the Revised Constraint Equation:

$w \;=\; \frac{p}{2}-l \;=\; \frac{p}{2}-\frac{p}{4} \;=\; \frac{p}{4}$ and thus $l=w$.

The corresponding rectangle is a square.

Verify that this CN leads to the absolute maximum of f on the feasible domain.

Method 1: Extreme Value Theorem (EVT) Method

f is continuous on the closed, bounded domain $\left[0,\frac{p}{2}\right]$, so the Extreme Value Theorem (EVT) applies. There exists an absolute maximum of f on $\left[0,\frac{p}{2}\right]$. Our candidates for l are $\frac{p}{4}$, which is the critical number (CN) of f in $\left(0,\frac{p}{2}\right)$, and the endpoints of the interval $\left[0,\frac{p}{2}\right]$.

l	$f(l)$	Comments
$a=0$	$f(0)=0$	
$\frac{p}{4}$	$f\left(\frac{p}{4}\right)=\frac{p^2}{16}$	There is an absolute maximum of f on $\left[0,\frac{p}{2}\right]$ at $\frac{p}{4}$.
$b=\frac{p}{2}$	$f\left(\frac{p}{2}\right)=0$	

Method 2a: General Second Derivative approach

$$f'(l)=\frac{p}{2}-2l \quad\Rightarrow$$

$$f''(l)=-2 \quad\Rightarrow\quad f''(l)<0,\quad \forall l\in\left(0,\frac{p}{2}\right), \text{including } l=\frac{p}{4}.$$

The graph of $y = f(l)$ is concave down on $\left(0, \frac{p}{2}\right)$, and f is continuous on $\left[0, \frac{p}{2}\right]$, so there must be an absolute maximum on $\left[0, \frac{p}{2}\right]$ at $l = \frac{p}{4}$, the sole critical number (CN) of f.

<u>Method 2b: Second Derivative Test approach</u>

- $f'\left(\frac{p}{4}\right) = 0$, so we may apply the Second Derivative Test at $l = \frac{p}{4}$.
- $f''\left(\frac{p}{4}\right) = -2 < 0$, so the graph of $y = f(l)$ is concave down "at" (actually, "on a neighborhood of") $l = \frac{p}{4}$, and there must be a local maximum there.
- Also, f is continuous on the feasible domain, $\left[0, \frac{p}{2}\right]$, and
- $\frac{p}{4}$ is the only critical number (CN) of f in $\left(0, \frac{p}{2}\right)$.

Therefore, the local maximum of f at $l = \frac{p}{4}$ must also be an absolute maximum on $\left[0, \frac{p}{2}\right]$.

<u>Method 3: First Derivative approach</u>

$f'(l) = \frac{p}{2} - 2l$

- $f'(l) > 0$ on the l-interval $\left(0, \frac{p}{4}\right)$,
- $f'(l) < 0$ on the l-interval $\left(\frac{p}{4}, \frac{p}{2}\right)$, and
- f is continuous on $\left[0, \frac{p}{2}\right]$.

Therefore, there must be an absolute maximum on $\left[0, \frac{p}{2}\right]$ at $l = \frac{p}{4}$.

l	$\left(0, \frac{p}{4}\right)$	$\boldsymbol{\frac{p}{4}}$	$\left(\frac{p}{4}, \frac{p}{2}\right)$
f' sign	+	0	–
f	↗		↘
f is cont. on $\left[0, \frac{p}{2}\right]$.		**A.Max.**	