

QUIZ ON CHAPTER 4 - SOLUTIONS

APPLICATIONS OF DERIVATIVES; MATH 150 – SPRING 2017 – KUNIYUKI

105 POINTS TOTAL, BUT 100 POINTS = 100%

1) Let $f(x) = x\sqrt{6x+1}$. Show all work, as in class. (16 points total)

a) What is the domain of f ? Write your answer in interval form (the form using parentheses and/or brackets). (3 points)

$$\text{Dom}(f) = \left[-\frac{1}{6}, \infty\right). \quad f(x) \text{ is real} \Leftrightarrow 6x+1 \geq 0 \Leftrightarrow 6x \geq -1 \Leftrightarrow x \geq -\frac{1}{6}.$$

b) Find and simplify $f'(x)$ as: $\frac{\text{a simplified polynomial in } x}{\sqrt{\text{a simplified polynomial in } x}}$.

Do not rationalize the denominator in your final answer. (9 points)

We need to use the **Product Rule** of Differentiation.

$$\begin{aligned} f'(x) &= D_x(x\sqrt{6x+1}) = [D_x(x)][\sqrt{6x+1}] + [x][D_x(\sqrt{6x+1})] \\ &= [1][\sqrt{6x+1}] + [x]\left(D_x\left[(6x+1)^{\frac{1}{2}}\right]\right) = \sqrt{6x+1} + x\left[\frac{1}{2}(6x+1)^{-\frac{1}{2}}(6)\right] \\ &= \sqrt{6x+1} + \frac{3x}{\sqrt{6x+1}} = \frac{\sqrt{6x+1}}{1} \cdot \frac{\sqrt{6x+1}}{\sqrt{6x+1}} + \frac{3x}{\sqrt{6x+1}} = \frac{6x+1}{\sqrt{6x+1}} + \frac{3x}{\sqrt{6x+1}} \\ &= \frac{6x+1+3x}{\sqrt{6x+1}} = \boxed{\frac{9x+1}{\sqrt{6x+1}}} \end{aligned}$$

c) Find and box in all critical number(s) of f . If there are none, write “NONE.” (4 points)

• The only x -value in $\text{Dom}(f)$, which is $\left[-\frac{1}{6}, \infty\right)$, that makes $f'(x)$ **undefined**

(DNE) is $-\frac{1}{6}$; it makes the denominator $\sqrt{6x+1}$ from b) equal to 0. Therefore, $-\frac{1}{6}$ is a critical number of f .

• Now, solve $f'(x) = 0$:

$$\frac{9x+1}{\sqrt{6x+1}} = 0$$

$$9x+1=0 \quad \left(\text{We require: } 6x+1 > 0, \text{ or } x > -\frac{1}{6} \right)$$

$$9x = -1 \quad \left(x > -\frac{1}{6} \right)$$

$$x = -\frac{1}{9} \quad \left(x > -\frac{1}{6} \right)$$

It is true that $-\frac{1}{9} > -\frac{1}{6}$, so $f'\left(-\frac{1}{9}\right) = 0$. Also, $-\frac{1}{9}$ is in $\text{Dom}(f)$, which is $\left[-\frac{1}{6}, \infty\right)$. Therefore, $-\frac{1}{9}$ is a critical number of f .

- The critical numbers of f are: $\boxed{-\frac{1}{6} \text{ and } -\frac{1}{9}}$.

2) Find the absolute maximum point (x, y) and the absolute minimum point (x, y) on the graph of $y = x^4 - 7x^3$, where x is restricted to the interval $[4, 6]$, in the usual xy -plane. Indicate which point is the absolute maximum point and which is the absolute minimum point. You may round off coordinates to two decimal places. (9 points)

Let $f(x) = x^4 - 7x^3$ on $[4, 6]$. f is a function that is continuous on the closed, bounded “domain” $[4, 6]$, so the **Extreme Value Theorem (EVT)** applies there. The EVT guarantees an absolute maximum and minimum for f on $[4, 6]$. We only check **critical numbers** in $(4, 6)$ and the **endpoints**, 4 and 6, for absolute extrema on $[4, 6]$.

- What are the critical number(s) of f in $(4, 6)$, if any?

$$f'(x) = 4x^3 - 21x^2$$

- $f'(x)$ is never undefined (DNE) on $(4, 6)$.

- Solve $f'(x) = 0$:

$$4x^3 - 21x^2 = 0$$

$$x^2(4x - 21) = 0$$

$$\cancel{x=0} \text{ or } 4x - 21 = 0$$

$$(0 \notin (4, 6)) \quad x = \frac{21}{4} \text{ or } 5.25$$

Since 0 is not in the interval $(4, 6)$, 5.25 is the only critical number in $(4, 6)$.

- We need to consider the x -value 5.25 and the endpoints of $[4, 6]$, 4 and 6.

x	$f(x)$	Answers
$a = 4$	$f(4) = -192$	$(4, -192)$ is the A.Max.Pt. on $[4, 6]$.
5.25	$f(5.25) = -\frac{64,827}{256} \approx -253.23$	$(5.25, -253.23)$ is the (approximate) A.Min.Pt. on $[4, 6]$.
$b = 6$	$f(6) = -216$	

- 3) State the Mean Value Theorem (MVT) for Derivatives, including the hypotheses and the conclusion. Write the conclusion using the algebraic notation we used in class; don't just refer to tangent lines or secant lines. (8 points)

If a function f is continuous on a closed, bounded interval $[a, b]$ and is differentiable on the open interval (a, b) , then $\exists c \in (a, b)$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$. (Assume $a < b$.)

Note: At such an x -value c , the tangent line to the graph of $y = f(x)$ is parallel to the secant line for the graph on $[a, b]$ - that is, the line through $(a, f(a))$ and $(b, f(b))$.

- 4) Let $s(t)$ be the height in feet (at time t in seconds) of a particle that is moving along a vertical line. If $s'(4) = -5 \frac{\text{ft}}{\text{sec}}$, and $s''(4) = -15 \frac{\text{ft}}{\text{sec}^2}$, what is the particle doing "at" (really, on a neighborhood of) $t = 4$ seconds? Box in one: (2 points)

a) The particle is rising and is speeding up.

b) The particle is rising and is slowing down.

c) The particle is falling and is speeding up.

d) The particle is falling and is slowing down.

$s'(4) = v(4) < 0$, so the particle's height is decreasing, and the particle is dropping.

$s''(4) = a(4) < 0$, so the particle is speeding up; the idea is that the velocity is getting more and more negative. (v is velocity; a is acceleration.) Think of a falling book.

- 5) Sketch the graph of $y = f(x)$, where $f(x) = x^6 + 6x^5 - 3$, in the usual xy -plane. (29 points)

- Find all critical numbers of f and label them CNs.
- Find all points at critical numbers.
Indicate these points on your graph.
- Find all inflection points (if any) and label them IPs.
Indicate these points on your graph.
- Classify all points at critical numbers as local maximum points, local minimum points, or neither.
- Find the y -intercept. You do not have to find x -intercepts.
- Have your graph show where f is increasing / decreasing and where the graph is concave up / concave down. Justify with work, as in class.
- You may round off any non-integers to five significant digits.
- Show all steps, as we have done in class.

Step 1: f is a nonconstant polynomial function. Therefore:

- Domain, $\text{Dom}(f) = \mathbb{R}$.
- f is continuous on \mathbb{R} .

- There are no horizontal asymptotes (HAs), vertical asymptotes (VAs), nor holes.
- f is nonlinear and polynomial, so the graph has no slant asymptotes (SAs).

For this f :

- $f(0) = -3$, the constant term of $x^6 + 6x^5 - 3$, so the y -intercept is at $(0, -3)$.
- f is neither even nor odd (x^6 and x^5 are present in descending powers form).

Step 2: Find $f'(x)$ and critical numbers (CNs) of f .

$$f'(x) = 6x^5 + 30x^4 = 6x^4(x+5)$$




f' is continuous on \mathbb{R} . In particular, it is never undefined (“DNE”).

The real solutions of $f'(x) = 0$ are 0 and -5 , which are in $\text{Dom}(f)$.

The CNs are -5 and 0 .

Step 3: Do a sign diagram for f' and classify the points at the CNs.

f is continuous on \mathbb{R} , so the First Derivative Test (1st DT) should apply wherever we have CNs. Both f and f' are continuous on \mathbb{R} , so we use just the CNs at -5 and 0 to divide the real number line into intervals. They are the only “fenceposts” where f' could change sign.

x	Test $x = -6$	-5	Test $x = -1$	0	Test $x = 1$
f' sign (see below)	$-$	0	$+$	0	$+$
f					
Classify Points at CNs (Use 1 st DT)		L.Min. Pt.		Neither	
Sub into $f(x)$ to get y		$(-5, f(-5))$ $(-5, -3128)$		$(0, f(0))$ $(0, -3)$	

The graph of $y = f'(x)$ is a “rising snake” in the long run, which explains the first and last signs. The multiplicity of the zero of f' at $x = -5$ is odd (1), so the signs differ in our “windows” around $x = -5$. The multiplicity of the zero at $x = 0$ is even (4), so the signs are the same in our “windows” around $x = 0$. Also:

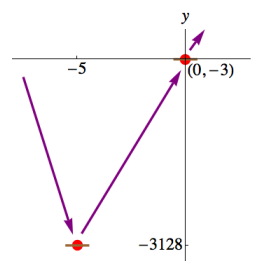
$$f'(x) = (6x^4)(x+5) \Rightarrow$$

$$f'(-6) = (+) (-) = -$$

$$f'(-1) = (+) (+) = +$$

$$f'(1) = (+) (+) = +$$

Step 4: Sketch a “skeleton graph” for $y = f(x)$ (optional).



Horizontal tangent line indicated in brown.

Step 5: Find $f''(x)$ and possible inflection numbers (PINs).

$$f'(x) = 6x^5 + 30x^4 \Rightarrow$$

$$f''(x) = 30x^4 + 120x^3 = 30x^3(x+4)$$

f'' is continuous on \mathbb{R} . In particular, it is never undefined (“DNE”).

The real solutions of $f''(x) = 0$ are 0 and -4 , which are in $\text{Dom}(f)$.

The PINs are -4 and 0 .

Step 6: Do a sign diagram for f'' and find inflection points (IPs).

f , f' , and f'' are continuous on \mathbb{R} , so we use just the PINs as “fenceposts” on the real number line where f'' could change sign.

x	Test $x = -5$	-4	Test $x = -1$	0	Test $x = 1$
f'' sign (see below)	+		−		+
f graph	CU (\cup)		CD (\cap)		CU (\cup)
Inflection Points (IPs)?		Yes, IP (f cont. here, concavity changes)		Yes, IP (f cont. here, concavity changes)	
Sub into $f(x)$ to get y		$(-4, f(-4))$ $(-4, -2051)$		$(0, f(0))$ $(0, -3)$	

The graph of $y = f''(x)$ is an “upward-opening bowl” in the long run, which explains the first and last signs. The multiplicity of the zero of f'' at $x = -4$ is odd (1), as is the multiplicity (3) of the zero at $x = 0$. Therefore, the signs alternate in the windows. Also:

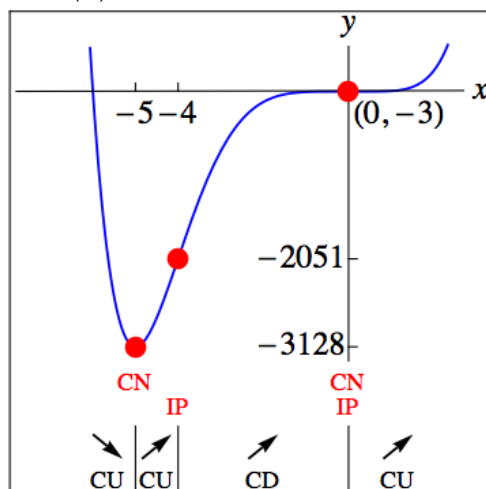
$$f''(x) = (30x^3)(x+4) \Rightarrow$$

$$f''(-5) = (-) \quad (-) = +$$

$$f''(-1) = (-) \quad (+) = -$$

$$f''(1) = (+) \quad (+) = +$$

Step 7: Sketch the graph of $y = f(x)$. (Remember your skeleton graph!)



“Zooming out (in the long run),” the graph is an “**upward-opening bowl**”; the leading term of $x^6 + 6x^5 - 3$ is x^6 , which has even degree and a positive leading coefficient.

6) The following is true of the polynomial function f :

$f(1)=6$, $f'(1)=0$, $f''(1)=-8$. The point $(1,6)$ on the graph of $y = f(x)$ in the usual xy -plane must be (Box in one:) (2 points)

a local minimum point

a local maximum point

neither

The Second Derivative Test can be applied here, because $f'(1) = 0$; 1 must be a critical number (CN). Because $f''(1) < 0$, the graph of $y = f(x)$ is **concave down** “at” (actually, “on a neighborhood of”) $x = 1$.

7) Let $f(x) = \frac{3x+5}{4x-7}$. Consider the graph of $y = f(x)$ in the usual xy -plane.

You do **not** have to show work. (4 points total)

a) Write the equation of the horizontal asymptote for the graph.

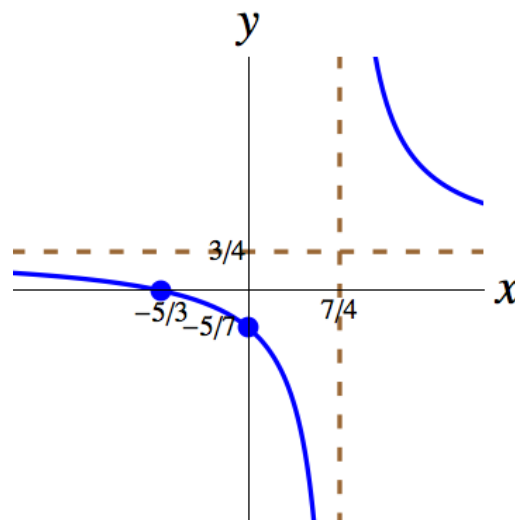
The numerator and denominator of $\frac{3x+5}{4x-7}$ are polynomials of the **same degree**, 1.

The horizontal asymptote (HA) is at: $y = \frac{3}{4}$, the **ratio of the leading coefficients**.

b) Write the equation of the vertical asymptote for the graph.

$\frac{3x+5}{4x-7}$ is **simplified and rational**. $\frac{7}{4}$ is the only **real zero of the denominator**.

Therefore, the only vertical asymptote (VA) is at: $x = \frac{7}{4}$.



8) We want to approximate a root (or zero) of $x^4 + 2x - 5$ using Newton's Method with $x_1 = 1.3$ as our "seed" (our first approximation). (10 points total)

a) Find x_2 , which is our second approximation using Newton's Method.

When rounding, use at least five significant digits; round off your answer to four decimal places.

Let $f(x) = x^4 + 2x - 5$. Then, $f'(x) = 4x^3 + 2$.

$$x_1 = 1.3 \Rightarrow$$

$$\begin{aligned} x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} = 1.3 - \frac{f(1.3)}{f'(1.3)} = 1.3 - \frac{(1.3)^4 + 2(1.3) - 5}{4(1.3)^3 + 2} \\ &= 1.3 - \frac{0.4561}{10.788} \approx 1.3 - 0.042278 \approx \boxed{1.2577} \end{aligned}$$

b) Find x_3 , which is our third approximation using Newton's Method.

When rounding, use at least five significant digits; round off your answer to four decimal places.

$$\begin{aligned} x_3 &= x_2 - \frac{f(x_2)}{f'(x_2)} \approx 1.2577 - \frac{f(1.2577)}{f'(1.2577)} \approx 1.2577 - \frac{(1.2577)^4 + 2(1.2577) - 5}{4(1.2577)^3 + 2} \\ &\approx 1.2577 - \frac{0.017521}{9.9578} \approx 1.2577 - 0.0017595 \approx \boxed{1.2559} \end{aligned}$$

Note 1: The real roots (zeros) of f are about -1.7027064 and 1.2559375 (the one we seem to be approaching). The two other complex roots are imaginary.

Note 2: $x_4 \approx 1.2559$. The last two iterates are equal to four decimal places, so we would take 1.2559 as our final approximation to four decimal places.

Note 3: We could have simplified the iteration formula (for $n \in \mathbb{Z}^+$):

$$x_{n+1} = x_n - \frac{x_n^4 + 2x_n - 5}{4x_n^3 + 2} = \frac{3x_n^4 + 5}{4x_n^3 + 2}$$

9) We need to make a (right circular) cylindrical metal soup can with a closed top and bottom and with a volume of 175 cubic inches. (25 points)

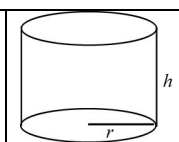
- Find the base radius and the height of such a cylinder that requires the least amount of metal, and box in these answers.
- You must round off your answers to at least three significant digits.
- Write your answers with appropriate units.
- Show all work, and verify that you are, in fact, finding the appropriate cylinder requiring the **absolute** minimum amount of metal, as in class.
- Do not use any shortcut precalculus formulas or methods when methods from calculus can be used instead.

Hints: The surface area of a closed right circular cylinder is given by $2\pi r^2 + 2\pi rh$, where r is the base radius and h is the height. If you forgot the formula for the volume, you can buy it from me for 3 points. You can't get negative points for this problem.

Step 1: Read the problem.

Step 2: Set up a diagram, a table, etc.

Let r = the base radius of the cylinder (in inches).
Let h = the height of the cylinder (in inches).



Step 3: Write the primary equation; this will lead to the objective function.

Minimize S , the surface area of the can.

$$S = \underbrace{\pi r^2}_{\text{Area of top lid}} + \underbrace{\pi r^2}_{\text{Area of bottom lid}} + \underbrace{2\pi rh}_{\text{Lateral surface area}}$$

$$S = 2\pi r^2 + 2\pi rh$$

Step 4: Write any relevant secondary (constraint) equations.

$$\text{Volume } V = \underbrace{\pi r^2}_{\text{Base area}} h$$

$$175 = \pi r^2 h$$

Step 5: Express S in terms of only one variable, say r .

Solve the secondary (constraint) equation for h in terms of r :

$$175 = \pi r^2 h$$

$$h = \frac{175}{\pi r^2} \quad (\text{Revised constraint equation})$$

Express S in terms of r . Incorporate the constraint into the primary equation:

$$S = 2\pi r^2 + 2\pi rh = 2\pi r^2 + 2\pi r \overset{(1)}{\left(\frac{175}{\cancel{\pi r^2} r} \right)} = 2\pi r^2 + \frac{350}{r}$$

We let $f(r) = 2\pi r^2 + \frac{350}{r}$. f is our **objective function** (to minimize).

Step 6: Write the “feasible domain” of f .

r can be any positive real number (in inches). Feasible domain: $(0, \infty)$ in inches.

Minimize f

Find critical numbers (CNs) of f in the feasible domain, $(0, \infty)$:

$$f(r) = 2\pi r^2 + \frac{350}{r} = 2\pi r^2 + 350r^{-1} \Rightarrow$$

$$f'(r) = 4\pi r - 350r^{-2} = 4\pi r - \frac{350}{r^2}$$

This is continuous on $(0, \infty)$; in particular, it is never undefined (“DNE”) there.

Find the zeros of $f'(r)$ in $(0, \infty)$ to get the CNs of f .

Solve $4\pi r - \frac{350}{r^2} = 0$ by multiplying both sides by r^2 .

$$4\pi r^3 - 350 = 0$$

$$4\pi r^3 = 350$$

$$r^3 = \frac{350}{4\pi} \text{ or } \frac{175}{2\pi}$$

$$r = \sqrt[3]{\frac{175}{2\pi}} \text{ (which is in the feasible domain, } (0, \infty))$$

$$r \approx 3.03 \text{ inches (Calculator: Divide by } \pi; \text{ don't multiply!)}$$

The sole CN of f is given by $r^* = \sqrt[3]{\frac{175}{2\pi}} \approx 3.03$ inches.

Verify that this CN (r^*) leads to the absolute minimum of f on the feasible domain.

Note: The Extreme Value Theorem (EVT) wouldn't apply here, because the feasible domain is not a closed, bounded interval.

Method 1a: Second Derivative Test approach

$f'(r^*) = 0$, so we may apply the Second Derivative Test at $r = r^*$.

$$f'(r) = 4\pi r - 350r^{-2} \Rightarrow$$

$$f''(r) = 4\pi + 700r^{-3} = 4\pi + \frac{700}{r^3} \Rightarrow$$

$$f''(r) > 0, \forall r \in (0, \infty), \text{ including } r^* \Rightarrow$$

$$f''(r^*) > 0$$

• Therefore, the graph of $y = f(r)$ is concave up “at” (actually, “on a neighborhood of”) r^* , so there must be a local minimum there.

• Also, f is continuous on the feasible domain, $(0, \infty)$, and

• There is only one critical number (CN) in $(0, \infty)$.

Therefore, the local minimum at $r = r^*$ must also be the absolute minimum.



Method 1b: General Second Derivative approach

Similarly, we could argue that the graph of $y = f(r)$ is concave up throughout the r -interval $(0, \infty)$, and f is continuous on $(0, \infty)$. The local minimum at $r = r^*$ must also be the absolute minimum.

Method 2: First Derivative approach

- $f'(r) < 0$ on the r -interval $(0, r^*)$,
- $f'(r) > 0$ on the r -interval (r^*, ∞) , and
- f is continuous on $(0, \infty)$.

Therefore, r^* must be the absolute minimum on $(0, \infty)$.

r	$(0, r^*)$	r^*	(r^*, ∞)
f' sign	–	0	+
f			
		A.Min. Pt.	

Answer the Questions

The base radius of the optimal cylinder is given by: $r \approx 3.03$ inches.

Now, use the [revised constraint equation](#) to find the corresponding height, h :

$$h = \frac{175}{\pi r^2} \approx \frac{175}{\pi (3.03)^2} \approx 6.07$$

The corresponding height is given by: $h \approx 6.07$ inches.

Note 1: It would have been more accurate to use:

$$h = \frac{175}{\pi r^2} = \frac{175}{\pi \left(\sqrt[3]{\frac{175}{2\pi}} \right)^2} = \sqrt[3]{\frac{700}{\pi}} \quad (\text{See Note 2 below.}) \approx 6.06$$

The height is then given by: $h \approx 6.06$ inches.

Note 2:

$$\frac{175}{\pi \left(\sqrt[3]{\frac{175}{2\pi}} \right)^2} = \frac{175}{\pi \left(\frac{175^{2/3}}{2^{2/3} \pi^{2/3}} \right)} = \frac{175^{1/3} \cdot 2^{2/3}}{\pi^{1/3}} = \frac{175^{1/3} \cdot 4^{1/3}}{\pi^{1/3}} = \left(\frac{700}{\pi} \right)^{1/3} = \sqrt[3]{\frac{700}{\pi}}$$

The optimal cylinder has base radius about 3.03 inches and height about 6.06 inches.

Note: The corresponding minimum surface area, $f(r^*) \approx 173 \text{ in}^2$.