QUIZ ON CHAPTER 5 - SOLUTIONS

INTEGRALS; MATH 150 – FALL 2016 – KUNIYUKI
105 POINTS TOTAL, BUT 100 POINTS = 100%

1) Evaluate the following integrals. Simplify as appropriate. (42 points total)

a) $\displaystyle\int \frac{5r^2 - 7r + \sqrt{r}}{r}\,dr$ (9 points)

$$= \int\left(\frac{5r^2}{r} - \frac{7r}{r} + \frac{\sqrt{r}}{r}\right)dr = \int\left(5r - 7 + \frac{1}{\sqrt{r}}\right)dr = \int\left(5r - 7 + r^{-1/2}\right)dr$$

$$= 5\left(\frac{r^2}{2}\right) - 7r + \frac{r^{1/2}}{1/2} + C = \boxed{\frac{5}{2}r^2 - 7r + 2\sqrt{r} + C, \text{ or } \frac{5r^2 - 14r + 4\sqrt{r}}{2} + C}$$

b) $\displaystyle\int \cot(\theta)\sec(\theta)\csc(\theta)\,d\theta$ (7 points)

$$= \int\left(\frac{\overset{(1)}{\cancel{\cos(\theta)}}}{\sin(\theta)}\right)\left(\frac{1}{\underset{(1)}{\cancel{\cos(\theta)}}}\right)\left(\frac{1}{\sin(\theta)}\right)d\theta = \int\frac{1}{\sin^2(\theta)}\,d\theta = \int\csc^2(\theta)\,d\theta$$

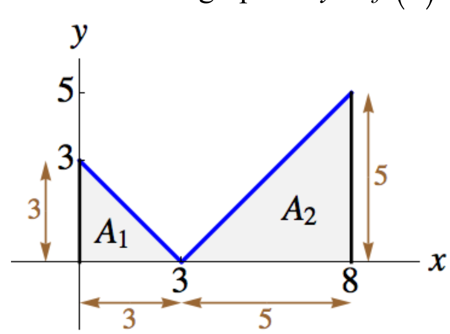
$$= \boxed{-\cot(\theta) + C, \text{ or } C - \cot(\theta)}$$

c) $\displaystyle\int_0^8 |x-3|\,dx$ (7 points)

(You may use the Fundamental Theorem of Calculus or a geometric argument.)

Method 1: Geometric argument

Let $f(x) = |x-3|$. The function f is nonnegative, so the value of the definite integral equals the total area under the graph of $y = f(x)$ from $x = 0$ to $x = 8$.



$$\int_0^8 |x-3|\,dx = A_1 + A_2 = \frac{1}{2}(3)(3) + \frac{1}{2}(5)(5) = \frac{9}{2} + \frac{25}{2} = \frac{34}{2} = \boxed{17}$$

Method 2: Fundamental Theorem of Calculus (FTC)

Observe that f is continuous on $[0,8]$, so the FTC, Part II applies.

By the definition of absolute value, $|x-3| = \begin{cases} -(x-3), \text{ or } 3-x, & \text{if } x \le 3 \\ x-3, & \text{if } x \ge 3 \end{cases}$.

(cont.)

$$\int_0^8 |x-3|\,dx = \int_0^3 (3-x)\,dx + \int_3^8 (x-3)\,dx = \left[3x - \frac{x^2}{2}\right]_0^3 + \left[\frac{x^2}{2} - 3x\right]_3^8$$

$$= \left(\left[3(3) - \frac{(3)^2}{2}\right] - [0]\right) + \left(\left[\frac{(8)^2}{2} - 3(8)\right] - \left[\frac{(3)^2}{2} - 3(3)\right]\right)$$

$$= \left(\frac{9}{2} - 0\right) + \left(8 - \left[-\frac{9}{2}\right]\right) = \frac{9}{2} + \frac{25}{2} = \frac{34}{2} = \boxed{17}$$

d) $\displaystyle\int \frac{\csc(x)\cot(x)}{\left[1+\csc(x)\right]^3}\,dx$ (7 points)

Let $u = 1+\csc(x) \Rightarrow$

$du = -\csc(x)\cot(x)\,dx \Rightarrow \left[\text{Can use: } \csc(x)\cot(x)\,dx = -\,du\right]$

$$\int \frac{\csc(x)\cot(x)}{\left[1+\csc(x)\right]^3}\,dx = -\int \frac{-\csc(x)\cot(x)}{\left[1+\csc(x)\right]^3}\,dx \;\; (\text{Compensation}) = -\int \frac{du}{u^3}$$

$$= -\int u^{-3}\,du = -\frac{u^{-2}}{-2} + C = \frac{1}{2u^2} + C = \boxed{\frac{1}{2\left[1+\csc(x)\right]^2} + C}$$

e) $\displaystyle\int_1^4 \frac{x}{\sqrt{3x^2+1}}\,dx$ (12 points)

Give an exact, simplified fraction as your answer.

The integrand is continuous on $[1,4]$, so the Fundamental Theorem of Calculus (FTC), Part II applies.

Let $u = 3x^2 + 1 \Rightarrow$

$du = 6x\,dx \Rightarrow \left(\text{Can use: } x\,dx = \frac{1}{6}du\right)$

Method 1 (Change the limits of integration.)

$x = 1 \Rightarrow u = 3(1)^2 + 1 = 4 \Rightarrow u = 4$

$x = 4 \Rightarrow u = 3(4)^2 + 1 = 49 \Rightarrow u = 49$

$$\int_1^4 \frac{x}{\sqrt{3x^2+1}}\,dx = \frac{1}{6}\int_1^4 \frac{6x}{\sqrt{3x^2+1}}\,dx \;\; (\text{Compensation}) \text{ or } \int_1^4 \left(\frac{1}{\sqrt{3x^2+1}}\right)\cdot(x\,dx)$$

$$= \frac{1}{6}\int_{x=1}^{x=4} \left(\frac{1}{\sqrt{3x^2+1}}\right)\cdot(6x\,dx) \text{ or } \int_{u=4}^{u=49} \left(\frac{1}{\sqrt{u}}\right)\cdot\left(\frac{1}{6}du\right) = \frac{1}{6}\int_{u=4}^{u=49} \frac{1}{\sqrt{u}}\,du$$

$$= \frac{1}{6}\int_4^{49} u^{-1/2}\,du = \frac{1}{6}\left[\frac{u^{1/2}}{1/2}\right]_4^{49} = 2\left(\frac{1}{6}\right)\left[\sqrt{u}\right]_4^{49} = \frac{1}{3}\left(\left[\sqrt{49}\right] - \left[\sqrt{4}\right]\right)$$

$$= \frac{1}{3}(7-2) = \boxed{\frac{5}{3}}$$

Method 2 (Work out the corresponding indefinite integral first.)

$$\int \frac{x}{\sqrt{3x^2+1}}\,dx = \frac{1}{6}\int \frac{6x}{\sqrt{3x^2+1}}\,dx \ \text{(Compensation)} \quad \text{or} \quad \int \left(\frac{1}{\sqrt{3x^2+1}}\right)\cdot(x\,dx)$$

$$= \frac{1}{6}\int \left(\frac{1}{\sqrt{3x^2+1}}\right)\cdot(6x\,dx) \quad \text{or} \quad \int \left(\frac{1}{\sqrt{u}}\right)\cdot\left(\frac{1}{6}\,du\right) = \frac{1}{6}\int \frac{1}{\sqrt{u}}\,du$$

$$= \frac{1}{6}\int u^{-1/2}\,du = \frac{1}{6}\left[\frac{u^{1/2}}{1/2}\right]+C = 2\left(\frac{1}{6}\right)\left[\sqrt{u}\right]+C = \frac{\sqrt{u}}{3}+C = \frac{\sqrt{3x^2+1}}{3}+C$$

Now, apply the FTC directly using our antiderivative (where $C=0$).

$$\int_1^4 \frac{x}{\sqrt{3x^2+1}}\,dx = \left[\frac{\sqrt{3x^2+1}}{3}\right]_1^4 = \frac{1}{3}\left[\sqrt{3x^2+1}\right]_1^4$$

$$= \frac{1}{3}\left(\left[\sqrt{3(4)^2+1}\right]-\left[\sqrt{3(1)^2+1}\right]\right) = \frac{1}{3}\left(\left[\sqrt{49}\right]-\left[\sqrt{4}\right]\right) = \frac{1}{3}(7-2) = \boxed{\frac{5}{3}}$$

2) Solve the second-order differential equation $\frac{d^2y}{dx^2}=\cos(2x)$ subject to the following initial conditions: $\frac{dy}{dx}=\frac{5}{4}$ when $x=\frac{\pi}{12}$, and $y=1$ when $x=0$. (14 points)

First Integration: $\left(\text{Find } \frac{dy}{dx}.\right)$

$$\frac{d^2y}{dx^2}=\cos(2x) \quad \Rightarrow \quad \int \frac{d^2y}{dx^2}\,dx = \int \cos(2x)\,dx$$

$$\frac{dy}{dx}=\frac{1}{2}\sin(2x)+C \quad \text{(by Guess-and-Check or a } u\text{-sub)}$$

Find C:

$$x=\frac{\pi}{12} \Rightarrow \frac{dy}{dx}=\frac{5}{4} \ \text{(This was given.)}, \text{ and } \frac{dy}{dx}=\frac{1}{2}\sin(2x)+C \Rightarrow$$

$$\frac{5}{4}=\frac{1}{2}\sin\left(2\cdot\frac{\pi}{12}\right)+C$$

$$\frac{5}{4}=\frac{1}{2}\sin\left(\frac{\pi}{6}\right)+C$$

$$\frac{5}{4}=\frac{1}{2}\left(\frac{1}{2}\right)+C$$

$$\frac{5}{4}=\frac{1}{4}+C$$

$$C=1 \Rightarrow$$

$$\frac{dy}{dx}=\frac{1}{2}\sin(2x)+1$$

Second Integration: (Find y.)

$$\frac{dy}{dx}=\frac{1}{2}\sin(2x)+1 \quad\Rightarrow\quad \int\frac{dy}{dx}\,dx=\int\left(\frac{1}{2}\sin(2x)+1\right)dx$$

$$y=-\frac{1}{4}\cos(2x)+x+D \quad (\text{by G-and-C or } u\text{-sub})$$

Find D:

$$x=0 \Rightarrow y=1, \text{ and } y=-\frac{1}{4}\cos(2x)+x+D \quad\Rightarrow$$

$$1=-\frac{1}{4}\cos(2\cdot 0)+0+D$$

$$1=-\frac{1}{4}\cos(0)+D$$

$$1=-\frac{1}{4}(1)+D$$

$$1=-\frac{1}{4}+D$$

$$D=\frac{5}{4} \quad\Rightarrow$$

$$\boxed{y=-\frac{1}{4}\cos(2x)+x+\frac{5}{4}, \text{ or } y=\frac{4x+5-\cos(2x)}{4}}$$

Note: If the Leibniz notation is uncomfortable, you may switch to Lagrange notation. Let $y=f(x)$. Then, $f'\left(\frac{\pi}{12}\right)=\frac{5}{4}$, and $f(0)=1$.

3) Assume that f is an everywhere continuous function on $\mathbb{R}$ such that $\int_2^5 f(x)\,dx=20$. Simplify and evaluate: $\int_5^8 f(x)\,dx+\int_8^2 f(x)\,dx$. (5 points)

$$\int_5^8 f(x)\,dx+\int_8^2 f(x)\,dx \;=\; \int_5^2 f(x)\,dx \;=\; -\int_2^5 f(x)\,dx \;=\; \boxed{-20}$$

4) Assume that f is an everywhere continuous **even** function on $\mathbb{R}$ such that $\int_0^{10} f(x)\,dx=30$. Evaluate $\int_{-10}^{10} 7f(x)\,dx$. (4 points)

$$\int_{-10}^{10} 7f(x)\,dx \;=\; 7\int_{-10}^{10} f(x)\,dx \;(\text{by Linearity}) \;=\; 7\left[2\int_0^{10} f(x)\,dx\right] \;(\text{by Even Property})$$

$$=\; 14\left[\int_0^{10} f(x)\,dx\right] \;=\; 14[30] \;=\; \boxed{420}$$

5) Let $f(x) = x^4 + 3$. (12 points total)

a) Find f_{av}, the average value of f on the interval $[-1, 2]$.

Write your answer as an exact, simplified fraction. (10 points)

$$f_{av} = \frac{\int_a^b f(x)\,dx}{b-a} = \frac{\int_{-1}^{2}\left(x^4+3\right)dx}{2-(-1)} = \frac{1}{3}\int_{-1}^{2}\left(x^4+3\right)dx$$

Note: The integrand is continuous on $[-1, 2]$, so the FTC applies.

$$= \frac{1}{3}\left[\frac{x^5}{5} + 3x\right]_{-1}^{2} = \frac{1}{3}\left(\left[\frac{(2)^5}{5} + 3(2)\right] - \left[\frac{(-1)^5}{5} + 3(-1)\right]\right)$$

$$= \frac{1}{3}\left(\left[\frac{32}{5} + 6\right] - \left[-\frac{1}{5} - 3\right]\right) = \frac{1}{3}\left(\frac{32}{5} + 6 + \frac{1}{5} + 3\right)$$

$$= \frac{1}{3}\left(\frac{33}{5} + 9\right) = \frac{1}{3}\left(\frac{33}{5} + \frac{45}{5}\right) = \frac{1}{3}\left(\frac{78}{5}\right) = \boxed{\frac{26}{5},\ \text{or } 5.2}$$

Note: The positive integer 78 is divisible by 3, because the digits of 78 add up to a multiple of 3. (7 + 8 = 15.)

b) True or False: There exists a real number z in the interval $(-1, 2)$ such that $f(z) = f_{av}$, where f_{av} is the [correct] answer to part a). Box in one: (Don't worry about the issue of open vs. closed intervals.) (2 points)

$\boxed{\text{True}}$ False

f is continuous on $[-1, 2]$, so the Mean Value Theorem (MVT) for Integrals applies.

6) Simplify $D_x\left(\int_5^x \frac{1}{t^2+1}\,dt\right)$. (3 points)

$$D_x\left(\int_5^x \underbrace{\frac{1}{t^2+1}}_{\substack{f(t),\text{ which is} \\ \text{everywhere} \\ \text{continuous on } \mathbb{R}}}\,dt\right) = \underbrace{\boxed{\frac{1}{x^2+1}}}_{f(x)}$$ by the Fundamental Theorem of Calculus (FTC), Part I.

7) We will approximate $\int_0^1 \tan(x)\,dx$ using two different methods.

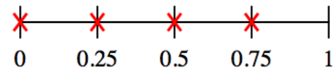
For part a) and part b), use a **regular** partition with $n = 4$ subintervals. Round off intermediate calculations to at least five significant digits, and round off your final answers to four significant digits. Show all work! Remember that x is to be measured in radians, as opposed to degrees. (15 points total)

Let $f(x) = \tan(x)$. f is continuous on $[0, 1]$, so it is integrable on $[0, 1]$. Let $a = 0,\ b = 1$.

a) Approximate $\int_0^1 \tan(x)\,dx$ by using a Left-hand Riemann Approximation (LRA).

Each $\Delta x_k = \Delta x = \dfrac{b-a}{n} = \dfrac{1-0}{4} = \dfrac{1}{4} = 0.25 \;\Rightarrow\;$ Partition: $\{0,\ 0.25,\ 0.5,\ 0.75,\ 1\}$.

For our LRA, we use: $w_1 = 0$, $w_2 = 0.25$, $w_3 = 0.5$, and $w_4 = 0.75$.



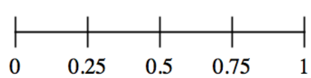
$$
\begin{aligned}
\int_0^1 \tan(x)\,dx &\approx \sum_{k=1}^{4}\left[f(w_k)\right]\left[\Delta x_k\right], \text{ or } \sum_{k=1}^{4}\left[f(w_k)\right]\left[\Delta x\right], \text{ or } \left[\Delta x\right]\left[\sum_{k=1}^{4} f(w_k)\right]\\
&\approx \sum_{k=1}^{4}\left[f(w_k)\right]\left[0.25\right]\\
&\approx \left[f(0)\right]\left[0.25\right]+\left[f(0.25)\right]\left[0.25\right]+\left[f(0.5)\right]\left[0.25\right]+\left[f(0.75)\right]\left[0.25\right]\\
&\approx 0.25\left[f(0) + f(0.25) + f(0.5) + f(0.75)\right]\\
&\approx 0.25\left[\tan(0) + \tan(0.25) + \tan(0.5) + \tan(0.75)\right]\\
&\approx 0.25\left[0 + 0.25534 + 0.54630 + 0.93160\right]\\
&\approx 0.25\left[1.73324\right]\\
&\approx \boxed{0.4333} \qquad \underline{\text{Note}}\text{: Exact value} = -\ln(\cos(1)) \approx 0.6156
\end{aligned}
$$

b) Approximate $\int_0^1 \tan(x)\,dx$ by using the Trapezoidal Rule.

Hint: The Trapezoidal Rule is given by:

$$
\begin{aligned}
\int_a^b f(x)\,dx &\approx \frac{b-a}{2n}\left[f(x_0)+2f(x_1)+2f(x_2)+\cdots+2f(x_{n-1})+f(x_n)\right], \text{ or}\\
&\approx \frac{1}{2}\Delta x\left[f(x_0)+2f(x_1)+2f(x_2)+\cdots+2f(x_{n-1})+f(x_n)\right]
\end{aligned}
$$

$\Delta x = \dfrac{b-a}{n} = \dfrac{1}{4} = 0.25 \;\Rightarrow\;$ Partition: $\{0,\ 0.25,\ 0.5,\ 0.75,\ 1\}$. (See work for a).)



<u>Note</u>: From the first form of the rule: $\dfrac{b-a}{2n} = \dfrac{1-0}{2(4)} = \dfrac{1}{8} = 0.125$. From the second:

$$
\begin{aligned}
\int_0^1 \tan x\,dx &\approx \frac{1}{2}(0.25)\left[f(0) + 2f(0.25) + 2f(0.5) + 2f(0.75) + f(1)\right]\\
&\approx (0.125)\left[\tan(0) + 2\tan(0.25) + 2\tan(0.5) + 2\tan(0.75) + \tan(1)\right]\\
&\approx (0.125)\left[0 + 2(0.25534) + 2(0.54630) + 2(0.93160) + 1.55741\right], \text{ or}\\
&\quad\ (0.125)\left[0 + 0.51068 + 1.09260 + 1.86319 + 1.55741\right]\\
&\approx \boxed{0.6280} \qquad \underline{\text{Note}}\text{: Exact value} = -\ln(\cos 1) \approx 0.6156\text{; compare to a).}
\end{aligned}
$$

8) Evaluate $\int \frac{x}{(x-10)^{50}}\,dx$. Do not leave negative exponents in your final answer.

Hint: Remember a trick discussed in class. (10 points)

$$\text{Let} \quad u = x-10 \quad \Rightarrow \quad x = u+10$$
$$du = dx$$

$$\begin{aligned}
\int \frac{x}{(x-10)^{50}}\,dx &= \int \frac{u+10}{u^{50}}\,du \\
&= \int \left(\frac{u}{u^{50}} + \frac{10}{u^{50}} \right) du \\
&= \int \left(\frac{1}{u^{49}} + \frac{10}{u^{50}} \right) du \\
&= \int \left(u^{-49} + 10u^{-50} \right) du \\
&= \frac{u^{-48}}{-48} + 10\left(\frac{u^{-49}}{-49} \right) + C \\
&= -\frac{1}{48u^{48}} - \frac{10}{49u^{49}} + C \\
&= \boxed{-\frac{1}{48(x-10)^{48}} - \frac{10}{49(x-10)^{49}} + C}
\end{aligned}$$

Algebra Challenge: Check this!

Note: This could be further simplified.

$$\begin{aligned}
&= -\left(\frac{1}{48(x-10)^{48}} + \frac{10}{49(x-10)^{49}} \right) + C \\
&= -\frac{49(x-10) + 48(10)}{(48)(49)(x-10)^{49}} + C \\
&= \boxed{-\frac{49x-10}{2352(x-10)^{49}} + C, \text{ or } \frac{10-49x}{2352(x-10)^{49}} + C, \text{ or } C - \frac{49x-10}{2352(x-10)^{49}}}
\end{aligned}$$