

QUIZ ON CHAPTER 5 - SOLUTIONS

INTEGRALS; MATH 150 – SPRING 2017 – KUNIYUKI

105 POINTS TOTAL, BUT 100 POINTS = 100%

1) Evaluate the following integrals. Simplify as appropriate. (47 points total)

a) $\int \frac{(w^2 + 4)^2}{w^2} dw$ (9 points)

$$= \int \frac{w^4 + 8w^2 + 16}{w^2} dw = \int \left(\frac{w^4}{w^2} + \frac{8w^2}{w^2} + \frac{16}{w^2} \right) dw = \int (w^2 + 8 + 16w^{-2}) dw$$
$$= \frac{w^3}{3} + 8w + 16 \left[\frac{w^{-1}}{-1} \right] + C = \boxed{\frac{w^3}{3} + 8w - \frac{16}{w} + C, \text{ or } \frac{w^4 + 24w^2 - 48}{3w} + C}$$

b) $\int \frac{[1 + \tan^2(\theta)] \tan(\theta)}{\sec(\theta)} d\theta$ (6 points)

$$= \int \frac{\sec^2(\theta) \tan(\theta)}{\sec(\theta)} d\theta \text{ (by a Pythagorean ID)} = \int \sec(\theta) \tan(\theta) d\theta = \boxed{\sec(\theta) + C}$$

c) $\int 3 \sec^2(x) \tan^7(x) dx$ (7 points)

$$\text{Let } u = \tan(x) \Rightarrow \\ du = \sec^2(x) dx$$

$$\int 3 \sec^2(x) \tan^7(x) dx = 3 \int \tan^7(x) \cdot \sec^2(x) dx = 3 \int u^7 du = 3 \left[\frac{u^8}{8} \right] + C$$
$$= \boxed{\frac{3}{8} \tan^8(x) + C}$$

d) $\int \frac{(2 + \sqrt{x})^6}{\sqrt{x}} dx$ (8 points)

$$\text{Let } u = 2 + \sqrt{x}, \text{ or } 2 + x^{1/2} \Rightarrow \\ du = \frac{1}{2} x^{-1/2} dx$$

$$du = \frac{1}{2\sqrt{x}} dx \Rightarrow \left(\text{Can use: } \frac{1}{\sqrt{x}} dx = 2 du \right)$$

$$\int \frac{(2 + \sqrt{x})^6}{\sqrt{x}} dx = \int (2 + \sqrt{x})^6 \cdot \frac{1}{\sqrt{x}} dx = 2 \int (2 + \sqrt{x})^6 \cdot \frac{1}{2\sqrt{x}} dx \text{ (Compensation)}$$
$$= 2 \int u^6 du = 2 \left[\frac{u^7}{7} \right] + C = \boxed{\frac{2}{7} (2 + \sqrt{x})^7 + C}$$

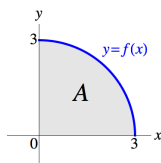
e) $\int_0^3 \sqrt{9-x^2} \, dx$ (5 points)

(Hint: Do not use the Fundamental Theorem of Calculus.)

Use geometry! Let $f(x) = \sqrt{9-x^2}$. The graph of $y = f(x)$ is the upper half of a circle of radius 3 centered at $(0, 0)$:

$$y = \sqrt{9-x^2} \Leftrightarrow y^2 = 9-x^2 \ (y \geq 0) \Leftrightarrow x^2 + y^2 = 9 \ (y \geq 0)$$

On the x -interval $[0, 3]$, we only pick up the quarter-circle seen below.



f is nonnegative on $[0, 3]$, so the value of the definite integral is equal to the area under the graph of $y = f(x)$ (and above the x -axis) from $x = 0$ to $x = 3$.

(We ignore units here.) We want the area A of the shaded quarter-circular region:

$$\int_0^3 \sqrt{9-x^2} \, dx = A = \frac{1}{4}\pi r^2 = \frac{1}{4}\pi(3)^2 = \boxed{\frac{9\pi}{4}}$$

You will learn an approach employing the FTC in Chapter 9 in Math 151.

You will use the trigonometric substitution (“trig sub”) $x = 3\sin(\theta)$.

f) $\int_0^2 \frac{x^2}{\sqrt{2x^3+9}} \, dx$ (12 points)

Give an exact, simplified fraction as your answer.

The integrand is continuous on $[0, 2]$, so the Fundamental Theorem of Calculus (FTC), Part II applies.

$$\text{Let } u = 2x^3 + 9 \Rightarrow$$

$$du = 6x^2 \, dx \Rightarrow \left(\text{Can use: } x^2 \, dx = \frac{1}{6} \, du \right)$$

Method 1 (Change the limits of integration.)

$$x = 0 \Rightarrow u = 2(0)^3 + 9 = 9 \Rightarrow u = 9$$

$$x = 2 \Rightarrow u = 2(2)^3 + 9 = 25 \Rightarrow u = 25$$

$$\begin{aligned} \int_0^2 \frac{x^2}{\sqrt{2x^3+9}} \, dx &= \frac{1}{6} \int_0^2 \frac{6x^2}{\sqrt{2x^3+9}} \, dx \quad (\text{by Compensation}) = \frac{1}{6} \int_9^{25} \frac{du}{\sqrt{u}} \\ &= \frac{1}{6} \int_9^{25} u^{-1/2} \, du = \frac{1}{6} \left[\frac{u^{1/2}}{1/2} \right]_9^{25} = \frac{1}{6} [2\sqrt{u}]_9^{25} = \frac{1}{3} [\sqrt{u}]_9^{25} = \frac{1}{3} (\sqrt{25} - \sqrt{9}) \\ &= \frac{1}{3} (5 - 3) = \boxed{\frac{2}{3}} \end{aligned}$$

Method 2 (Work out the corresponding indefinite integral first.)

$$\begin{aligned}\int \frac{x^2}{\sqrt{2x^3+9}} dx &= \frac{1}{6} \int \frac{6x^2}{\sqrt{2x^3+9}} dx \text{ (by Compensation)} = \frac{1}{6} \int \frac{du}{\sqrt{u}} \\ &= \frac{1}{6} \int u^{-1/2} du = \frac{1}{6} \left[\frac{u^{1/2}}{1/2} \right] + C = \frac{1}{6} [2\sqrt{u}] + C = \frac{1}{3} [\sqrt{u}] + C \\ &= \boxed{\frac{1}{3} \sqrt{2x^3+9} + C}\end{aligned}$$

Now, apply the FTC directly using our antiderivative (where $C = 0$).

$$\begin{aligned}\int_0^2 \frac{x^2}{\sqrt{2x^3+9}} dx &= \left[\frac{1}{3} \sqrt{2x^3+9} \right]_0^2 = \frac{1}{3} \left[\sqrt{2(2)^3+9} - \sqrt{2(0)^3+9} \right] \\ &= \frac{1}{3} [\sqrt{25} - \sqrt{9}] = \frac{1}{3} [5 - 3] = \boxed{\frac{2}{3}}\end{aligned}$$

- 2) An astronaut crawls to the edge of a cliff on planet Dork. The edge lies 33 feet above a lake. The astronaut throws down a rock at 30 feet per second.

The acceleration function for the rock is given by $a(t) = -6 \frac{\text{ft}}{\text{sec}^2}$, which is the

[signed] gravitational constant for Dork. The variable t represents time in seconds after the rock was thrown. Find the height function [rule] $s(t)$ for the height of the rock above the lake. $s(t)$ is measured in feet. [Note: Your $s(t)$ rule will only be relevant between the time the rock is thrown and the time the rock hits the lake.] Show all work, as in class. (9 points)

First Integration:

$$\begin{aligned}a(t) &= -6 \Rightarrow \\ \int a(t) dt &= \int -6 dt \\ v(t) &= -6t + C\end{aligned}$$

Find C:

Use: $v(0) = -30 \left(\frac{\text{ft}}{\text{s}} \right)$. **WARNING:** $v(0) < 0$, because the rock is being thrown down at time $t = 0$.

$$\begin{aligned}v(t) &= -6t + C \Rightarrow \\ v(0) &= -6(0) + C \\ -30 &= C \\ C &= -30 \Rightarrow \\ v(t) &= -6t - 30 \left(\text{in } \frac{\text{ft}}{\text{s}} \right)\end{aligned}$$

Second Integration:

$$v(t) = -6t - 30 \Rightarrow$$

$$\int v(t) dt = \int (-6t - 30) dt$$

$$s(t) = -6 \left[\frac{t^2}{2} \right] - 30t + D$$

$$s(t) = -3t^2 - 30t + D$$

Find D:

$$\text{Use: } s(0) = 33 \text{ (ft).}$$

$$s(t) = -3t^2 - 30t + D \Rightarrow$$

$$s(0) = -3(0)^2 - 30(0) + D$$

$$33 = D$$

$$D = 33 \Rightarrow$$

$$\boxed{s(t) = -3t^2 - 30t + 33 \text{ (in feet)}}$$

3) Assume that f is an everywhere continuous function on \mathbb{R} such that

$$\int_{10}^{20} f(x) dx = 100. \text{ Evaluate: } \int_{20}^{10} [3f(x) - 1] dx. \text{ (5 points)}$$

$$\begin{aligned} \int_{20}^{10} [3f(x) - 1] dx &= 3 \int_{20}^{10} f(x) dx - \int_{20}^{10} 1 dx \text{ (Linearity)} = -3 \int_{10}^{20} f(x) dx + \int_{10}^{20} 1 dx \\ &= -3(100) + [x]_{10}^{20} = -300 + (20 - 10) = -300 + 10 = \boxed{-290} \end{aligned}$$

4) Evaluate: $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan(x) dx$. **Answer only is fine.** (2 points)

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan(x) dx = \boxed{0}. \text{ Why? Let } f(x) = \tan(x). \text{ } f \text{ is } \mathbf{odd} \text{ and } \mathbf{continuous} \text{ on } \left[-\frac{\pi}{4}, \frac{\pi}{4} \right],$$

and the limits of integration, $-\frac{\pi}{4}$ and $\frac{\pi}{4}$, are **opposites**.

5) Simplify: $D_x \left(\int_{\pi}^x \sin(t^2) dt \right)$. (2 points)

$$D_x \left(\int_{\pi}^x \sin(t^2) dt \right) = \boxed{\sin(x^2)} \text{ by the Fundamental Theorem of Calculus (FTC), Part I.}$$

6) For parts a), b), c), and d), let $f(x) = x^4$. (30 points total)

f is continuous on $[a = 2, b = 4]$, so it is integrable on $[2, 4]$.

a) Approximate $\int_2^4 x^4 dx$ by using a Right-hand Riemann Approximation (RRA) based on the partition $\{2.0, 2.7, 3.0, 3.6, 4.0\}$. Round off calculations to at least five significant digits. (10 points)

For our RRA, we use: $w_1 = 2.7$, $w_2 = 3.0$, $w_3 = 3.6$, and $w_4 = 4.0$.

The subinterval widths are:

$$\Delta x_1 = 2.7 - 2.0 = 0.7 \quad \Delta x_3 = 3.6 - 3.0 = 0.6$$

$$\Delta x_2 = 3.0 - 2.7 = 0.3 \quad \Delta x_4 = 4.0 - 3.6 = 0.4$$

$$\begin{aligned} \int_2^4 f(x) dx &\approx \sum_{k=1}^4 [f(w_k)] [\Delta x_k] \\ &\approx [f(2.7)][0.7] + [f(3.0)][0.3] + [f(3.6)][0.6] + [f(4.0)][0.4] \\ &\approx (2.7)^4(0.7) + (3.0)^4(0.3) + (3.6)^4(0.6) + (4.0)^4(0.4) \\ &\approx (53.1441)(0.7) + (81)(0.3) + (167.9616)(0.6) + (256)(0.4) \\ &\approx 37.20087 + 24.3 + 100.77696 + 102.4 \\ &\approx \boxed{264.67783 \quad (\approx 264.68)} \end{aligned}$$

b) Approximate $\int_2^4 x^4 dx$ by using the Trapezoidal Rule. Use a regular partition with $n = 4$ subintervals. Round off calculations to at least five significant digits. Hint: The Trapezoidal Rule is given by: (12 points)

$$\begin{aligned} \int_a^b f(x) dx &\approx \frac{b-a}{2n} [f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)], \text{ or} \\ &\approx \frac{1}{2} \Delta x [f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)] \end{aligned}$$

$$\Delta x = \frac{b-a}{n} = \frac{4-2}{4} = \frac{2}{4} = \frac{1}{2} = 0.5 \Rightarrow \text{Partition: } \{2.0, 2.5, 3.0, 3.5, 4.0\}$$

$$\begin{aligned} \int_2^4 f(x) dx &\approx \frac{1}{2}(0.5) [f(2.0) + 2f(2.5) + 2f(3.0) + 2f(3.5) + f(4.0)] \\ &\approx 0.25 [(2.0)^4 + 2(2.5)^4 + 2(3.0)^4 + 2(3.5)^4 + (4.0)^4] \\ &\approx 0.25 [16 + 2(39.0625) + 2(81) + 2(150.0625) + 256] \\ &\approx 0.25 [16 + 78.125 + 162 + 300.125 + 256] \\ &\approx 0.25 [812.25] \\ &\approx \boxed{203.0625 \quad (\approx 203.06)} \end{aligned}$$

b) using fractions:

$$\begin{aligned} \int_2^4 f(x) dx &\approx \frac{1}{4} \left[f(2) + 2f\left(\frac{5}{2}\right) + 2f(3) + 2f\left(\frac{7}{2}\right) + f(4) \right] \\ &\approx \frac{1}{4} \left[(2)^4 + 2\left(\frac{5}{2}\right)^4 + 2(3)^4 + 2\left(\frac{7}{2}\right)^4 + (4)^4 \right] \approx \frac{1}{4} \left[16 + 2\left(\frac{625}{16}\right) + 2(81) + 2\left(\frac{2401}{16}\right) + 256 \right] \\ &\approx \frac{1}{4} \left[\frac{256 + 1250 + 2592 + 4802 + 4096}{16} \right] \approx \frac{12,996}{64} \approx \frac{3249}{16} \approx \boxed{203.0625 \quad (\approx 203.06)} \end{aligned}$$

- c) Find the exact value of $\int_2^4 x^4 dx$ by applying the Fundamental Theorem of Calculus. (5 points)

$$\int_2^4 x^4 dx = \left[\frac{x^5}{5} \right]_2^4 = \frac{(4)^5}{5} - \frac{(2)^5}{5} = \frac{1024}{5} - \frac{32}{5} = \boxed{\frac{992}{5}, \text{ or } 198\frac{2}{5}, \text{ or } 198.4}$$

Think about it: Why do we get overestimates in a) and b)? The reasons differ.
Think: f' and f'' .

- d) Use part c) to find f_{av} , the average value of f on the x -interval $[2, 4]$, where $f(x) = x^4$. (3 points)

$$f_{av} = \frac{\int_a^b f(x) dx}{b-a} = \frac{\int_2^4 x^4 dx}{4-2} = \frac{\frac{992}{5}}{2}, \text{ or } \frac{198.4}{2} = \boxed{\frac{496}{5}, \text{ or } 99\frac{1}{5}, \text{ or } 99.2}$$

- 7) Evaluate $\int x(x+5)^{200} dx$. Hint: Remember a trick discussed in class.
(10 points)

$$\text{Let } u = x+5 \Rightarrow x = u-5 \\ du = dx$$

$$\begin{aligned} \int x(x+5)^{200} dx &= \int (u-5)u^{200} du \quad (\text{Distribute; do NOT integrate factor-by-factor.}) \\ &= \int (u^{201} - 5u^{200}) du \\ &= \frac{u^{202}}{202} - 5 \left(\frac{u^{201}}{201} \right) + C \\ &= \boxed{\frac{1}{202}(x+5)^{202} - \frac{5}{201}(x+5)^{201} + C} \end{aligned}$$

Algebra Challenge: Check this!

This could be further simplified:

$$\begin{aligned} &= \frac{201(x+5)^{202} - 202 \cdot 5(x+5)^{201}}{202 \cdot 201} + C \\ &= \frac{(x+5)^{201} [201(x+5) - 1010]}{40,602} + C \\ &= \frac{(x+5)^{201} [201x + 1005 - 1010]}{40,602} + C \\ &= \boxed{\frac{(x+5)^{201} [201x - 5]}{40,602} + C} \end{aligned}$$