1) Find the area of the region \( R \) bounded by the graphs of \( x = y^2 - 4 \) and \( x + y = 2 \). Sketch and shade in the region \( R \), and identify any intersection points and \( x \)-intercepts. Evaluate your integral completely, and write your answer as an exact fraction or an exact mixed number in simplest form, together with appropriate units. Show all work. (25 points)

- The given equation \( x = y^2 - 4 \) is solved for \( x \) in terms of \( y \). We hope to do a “\( dy \) scan.”
- Find the \( y \)-coordinates of the intersection points of the corresponding graphs by solving the following system for \( y \); they will be the limits of integration.

\[
\begin{cases}
  x = y^2 - 4 \\
  x + y = 2 \quad \text{(for \( dy \) scan, solve for \( x \) in terms of \( y \))} \\
  x = y^2 - 4 \quad \text{(parabola opening right; leading coefficient on right is positive)} \\
  x = 2 - y \quad \text{(line)}
\end{cases}
\]

We will equate the expressions for \( x \):

\[
y^2 - 4 = 2 - y
\]

\[
y^2 + y - 6 = 0
\]

\[
(y + 3)(y - 2) = 0
\]

\[
y + 3 = 0 \quad \text{or} \quad y - 2 = 0
\]

\[
y = -3 \quad \text{or} \quad y = 2
\]

- Our limits of integration will apparently be \( a = -3 \) and \( b = 2 \).

- We will use the equation \( x = 2 - y \) to determine the \( x \)-coordinates of the intersection points. (We could also use the equation \( x = y^2 - 4 \).)

\[
y = -3 \quad \Rightarrow \quad x = 2 - (-3) \quad \Rightarrow \quad x = 5 \quad \Rightarrow \quad \text{Point} \ (5, -3)
\]

\[
y = 2 \quad \Rightarrow \quad x = 2 - (2) \quad \Rightarrow \quad x = 0 \quad \Rightarrow \quad \text{Point} \ (0, 2)
\]

The point \((0, 2)\) is the sole \( y \)-intercept of the line. It is also one of two \( y \)-intercepts for the parabola; the other is at \((0, -2)\), since 2 and \(-2\) are real zeros of \( y^2 - 4 \).

- Which forms the right boundary of \( R \): the parabola or the line? Which forms the left?

  - The parabola opens to the right, so it forms the left boundary of \( R \).
  - Test a \( y \)-value in the interval \((-3, 2)\), say \( y = 0 \); we then identify \( x \)-intercepts.

Graph of parabola: \( x = (0)^2 - 4 \quad \Rightarrow \quad x = -4 \quad \Rightarrow \quad \text{x-int. is} \ (-4, 0) \) [.]

Graph of line: \( x = 2 - (0) \quad \Rightarrow \quad x = 2 \quad \Rightarrow \quad \text{x-int. is} \ (2, 0) \).
2 > -4, so the line forms the right boundary of R.

• We will do a “dy scan,” because the region R is well suited to that, and the given equations are easily solved for x in terms of y.
• We’re not revolving R about any axis, so the fact that the x- and y-axes pass through the interior of R does not pose a problem.
• A, the area of R, is given by:

\[
A = \int_{-3}^{2} \left[ \frac{(2-y) - (y^2 - 4)}{x_{\text{right}(y)}} \right] dy = \int_{-3}^{2} \left[ 2 - y - y^2 + 4 \right] dy = \int_{-3}^{2} \left[ -y^2 - y + 6 \right] dy
\]

\[
= \left[ -\frac{y^3}{3} - \frac{y^2}{2} + 6y \right]_{-3}^{2} = \left[ -\frac{2}{3} - \frac{2}{2} + 6(2) \right] - \left[ -\frac{(-3)^3}{3} - \frac{(-3)^2}{2} + 6(-3) \right]
\]

\[
= \left[ -\frac{8}{3} - 2 + 12 \right] - \left[ \frac{9}{2} - 18 \right] = \left[ -\frac{8}{3} + 10 \right] - \left[ \frac{9}{2} - 9 \right] = -\frac{8}{3} + 10 + \frac{9}{2} + 9
\]

\[
= \frac{9}{2} - \frac{8}{3} + 19 = \frac{27}{6} - \frac{16}{6} + 19 = \frac{11}{6} + 19 = \frac{5}{6} + 19 = \boxed{\frac{205}{6} \text{ m}^2, \text{ or } \frac{125}{6} \text{ m}^2}
\]

2) Using the Disk Method, as we have discussed in class, find the volume of a right circular cone of altitude h and base radius r. Show all work. (20 points)

• Such a cone is obtained by revolving the triangular region R about the x-axis.
• We are revolving R about a horizontal axis, so the Disk Method requires a “dx scan.”
• Observe that the axis of revolution does not pass through the interior of R.
• Find the function $f$ such that $y = f(x)$ models the slanted line segment.

$$y = mx + b$$

$$y = \left(\frac{\text{rise}}{\text{run}}\right)x + 0$$

$$y = \frac{r}{h}x \quad (0 \leq x \leq h)$$

• We will apply the Disk Method to find the volume ($V$) of the cone.

$$V = \int_0^h \pi \left(\text{radius}\right)^2 \, dx = \int_0^h \pi \left(\frac{r}{h}x\right)^2 \, dx = \pi \int_0^h \frac{r^2}{h^2} x^2 \, dx \quad \text{(Key: } \frac{r^2}{h^2} \text{ is a constant)}$$

$$= \frac{\pi r^2}{h^2} \int_0^h x^2 \, dx = \frac{\pi r^2}{h^2} \left[ \frac{x^3}{3} \right]_0^h = \frac{\pi r^2}{h^2} \left[ \frac{(h)^3}{3} - [0] \right] = \frac{\pi r^2}{h^2} \left( \frac{h^3}{3} \right) = \frac{1}{3} \pi r^2 h \text{ (in m$^3$)}$$

3) Let $B$ be the region bounded by the graphs of $y = x^3$, $y = -x$, and $x = 2$.

**Sketch and shade in the region $B$.** Find the volume of the solid that has $B$ as its base if every cross section by a plane perpendicular to the $x$-axis is a square. **Evaluate** your integral completely, and give appropriate units. You may **round off** your final answer to four significant digits. Show all work. (14 points)

• The cross-sections are perpendicular to the $x$-axis, so we will integrate with respect to $x$ (“$dx$ scan”). The first two given equations are already solved for $y$ in terms of $x$.

![Graph of the region B](image)

• Fix a generic $x$-value in the interval $[0, 2]$. The cross-section at $x$ is a square (which degenerates to a point at $x = 0$) whose side length is:

$$s(x) = y_{\text{top}}(x) - y_{\text{bottom}}(x)$$

$$= x^3 - (-x)$$

$$= x^3 + x$$
• The area of the square is: \[ A(x) = \left[ s(x) \right]^2 = (x^3 + x)^2. \]

\[ x^3 + x \]

\[ x^3 + x \]

• \( V \), the volume of the solid, is given by:

\[ V = \int_0^2 A(x) \, dx = \int_0^2 \left( x^3 + x \right)^2 \, dx = \int_0^2 \left( x^6 + 2x^4 + x^2 \right) \, dx = \left[ \frac{x^7}{7} + \frac{2x^5}{5} + \frac{x^3}{3} \right]_0^2 \]

\[ = \left( \frac{2^7}{7} + \frac{2(2)^5}{5} + \frac{(2)^3}{3} \right) - \left[ 0 \right] = \frac{128}{7} + \frac{64}{5} + \frac{8}{3} = \frac{1920}{105} + \frac{1344}{105} + \frac{280}{105} \]

\[ = \frac{3544}{105} \text{ m}^3, \text{ or } \frac{79}{105} \text{ m}^3 = 33.75 \text{ m}^3 \]

4) The region \( R \) is bounded by the graphs of \( y = 0 \), \( y = x \), and \( 5x + 2y = 10 \).

**Sketch and shade in the region** \( R \), and **identify any intersection points and intercepts. Set up the integral** for the volume of the solid generated if \( R \) is revolved about the \( y \)-axis. **Do not evaluate.** Use the Washer Method.

(15 points)

• We are revolving \( R \) about a vertical axis (the \( y \)-axis), so we must do a “\( dy \) scan” to use the Washer Method. Solve for \( x \) in terms of \( y \) in the last two given equations.

\[ y = x \quad 5x + 2y = 10 \]
\[ x = y \quad 5x = 10 - 2y \]
\[ x = 2 - \frac{2}{5}y \]

• Find the intersection point shared by the corresponding lines:

Find the \( y \)-coordinate by solving the system:

\[ \begin{aligned} x &= y \\ x &= 2 - \frac{2}{5}y \end{aligned} \]

for \( y \).

We will equate the expressions for \( x \):

\[ y = 2 - \frac{2}{5}y \]
\[ \frac{7}{5}y = 2 \]
\[ y = \frac{10}{7} \]

Then, \( x = y \) \( \Rightarrow \) \( x = \frac{10}{7} \). The intersection point is: \( \left( \frac{10}{7}, \frac{10}{7} \right) \).
• The graph of \( x = 2 - \frac{2}{5}y \) is the red line that forms the right boundary of \( R \).
  - It forms the outer boundary of \( R \) relative to the axis of revolution.
  - Looking at the original equation, \( 5x + 2y = 10 \), observe that the \( x \)-intercept is at \((2, 0)\), and the \( y \)-intercept is at \((0, 5)\).

• The graph of \( x = y \) is the blue line that forms the left boundary of \( R \).
  - It forms the inner boundary of \( R \) relative to the axis of revolution.
  - The point \((0, 0)\) is both the sole \( x \)-intercept and the sole \( y \)-intercept.

• The region \( R \) is well-suited to a “\( dy \) scan.”
• The \( y \)-axis (the axis of revolution) does not pass through the interior of \( R \).

Fix a generic \( y \)-value in the interval \([0, \frac{10}{7}]\).

Find the radii \( r_{\text{out}}(y) \) and \( r_{\text{in}}(y) \) of the corresponding washer:

\[
r_{\text{out}}(y) = x_{\text{right}}(y) - x_{\text{left}}(y) = \left(2 - \frac{2}{5}y\right) - (0) = 2 - \frac{2}{5}y
\]

\[
r_{\text{in}}(y) = x_{\text{right}}(y) - x_{\text{left}}(y) = (y) - (0) = y
\]

\( V \), the volume of the solid, is given by:

\[
V = \int_{0}^{\frac{10}{7}} \left( \pi \left[ (2 - \frac{2}{5}y) - y \right]^2 - \pi y^2 \right) dy
\]

\[
= \int_{0}^{\frac{10}{7}} \left[ \pi \left(2 - \frac{2}{5}y\right)^2 - \pi y^2 \right] dy, \quad \text{or} \quad \pi \int_{0}^{\frac{10}{7}} \left[ \left(2 - \frac{2}{5}y\right)^2 - (y)^2 \right] dy, \quad \text{or} \quad \pi \int_{0}^{\frac{10}{7}} \left[ 4 - \frac{8}{5}y - \frac{21}{25}y^2 \right] dy \quad (\text{in m}^3)
\]
Note 1: If you did not know that the graph of \( x = 2 - \frac{2}{5} y \) lay to the right of the graph of \( x = y \), then test (say) \( y = 1 \) to see that \( 2 - \frac{2}{5} y \geq y \) on the \( y \)-interval \([0, \frac{10}{7}]\).

Note 2: The volume is: \( \frac{160\pi}{49} \) m\(^3\) \( \approx 10.258 \) m\(^3\). Check this for yourself!

5) The region \( R \) lies only in Quadrant I and is bounded by the graphs of \( y = x^3 \), \( y = 3x^3 \), and \( x = 1 \). **Sketch and shade in the region \( R \). Set up the integral** for the volume of the solid generated if \( R \) is revolved about the line \( x = 2 \). **Do not evaluate.** Use the Cylindrical Shell (Cylinder) Method. (15 points)

- We are revolving \( R \) about a vertical axis (\( x = 2 \)), so we must do a “\( dx \) scan” to use the Cylindrical Shell (Cylinder) Method.
- We already have the first two given equations solved for \( y \) in terms of \( x \).
- The graphs of \( y = x^3 \) and \( y = 3x^3 \) intersect at \((0, 0)\) but at no other point:
  
  Consider the system:
  \[
  \begin{align*}
  y &= x^3 \\
  y &= 3x^3
  \end{align*}
  \]
  
  Equate the expressions for \( y \) and solve for \( x \):
  
  \[
  x^3 = 3x^3 \\
  0 = 2x^3 \\
  x = 0
  \]
  
  \( x = 0 \implies y = 0 \implies (0, 0) \) is the [only] intersection point.

- Which graph forms the **top** boundary of \( R \) on the \( x \)-interval \([0, 1]\): \( y = x^3 \) or \( y = 3x^3 \)?
  
  The related functions are continuous on \([0, 1]\), with graphs only intersecting at \((0, 0)\). Test any \( x \)-value in the interval \((0, 1)\). If we test \( x = \frac{1}{2} \), for example:
  
  \[
  y = x^3 \implies y = \left( \frac{1}{2} \right)^3 = \frac{1}{8} \quad \text{(bottom)}
  \]
  
  \[
  y = 3x^3 \implies y = 3 \left( \frac{1}{2} \right)^3 = \frac{3}{8} \quad \text{(top)}
  \]
  
  \( \frac{3}{8} > \frac{1}{8} \), so the graph of \( y = 3x^3 \) is on **top**; it forms the top boundary of \( R \).

- The “corners” of \( R \) are at \((0, 0)\), \((1, 1)\), and \((1, 3)\). The last two lie on the line \( x = 1 \):
  
  On the graph of \( y = x^3 \): \( x = 1 \implies y = (1)^3 = 1 \implies \text{Point}(1, 1) \)
  
  On the graph of \( y = 3x^3 \): \( x = 1 \implies y = 3(1)^3 = 3 \implies \text{Point}(1, 3) \)
• Observe that the axis of revolution \((x = 2)\) does not pass through the interior of \(R\).

Fix a generic \(x\)-value in the interval \([0, 1]\).

Find the radius \(r(x)\) and the height \(h(x)\) of the corresponding cylinder:

\[
\begin{align*}
\text{radius } r(x) &= x_{\text{right}}(x) - x_{\text{left}}(x) \\
&= (2) - (x) \\
&= 2 - x \\
\text{height } h(x) &= y_{\text{top}}(x) - y_{\text{bottom}}(x) \\
&= (3x^3) - (x^3) \\
&= 2x^3
\end{align*}
\]

\(V\), the volume of the solid, is given by:

\[
V = \int_0^1 2\pi \left[ r(x) \right] h(x) \, dx
\]

\[
= \left[ 2\pi \left( 2 - x \right) \left( 2x^3 \right) \right] \, dx, \text{ or } 4\pi \int_0^1 \left( 2x^3 - x^4 \right) \, dx \quad \text{[in m}^3\text{]}
\]

Note: The volume is \(\frac{6\pi}{5} \text{ m}^3 = 3.770 \text{ m}^3\). Check this for yourself!

6) **Set up the integral** for the arc length of the graph of \(x = 2y^3 - 3y + 5\) from \((-5, -2)\) to \((4, 1)\). Your final variable of integration must be \(x\) or \(y\), as used in this problem. **You do not have to sketch a graph. Do not evaluate.** (6 points)

Let \(f(y) = 2y^3 - 3y + 5\). Then, \(f\) is continuous on the \(y\)-interval \([-2, 1]\).

\(f'(y) = 6y^2 - 3\), so \(f'\) is also continuous there.

(The \(x\)- and \(y\)-axes are scaled differently.)
Arc length, \( L = \int_{y=-2}^{y=1} ds \)
\[ = \int_{-2}^{1} \sqrt{1 + [f'(y)]^2} \, dy \]
\[ = \int_{-2}^{1} \sqrt{1 + [6y^2 - 3]^2} \, dy, \text{ or} \]
\[ \int_{-2}^{1} \sqrt{36y^4 - 36y^2 + 10} \, dy \text{ (in meters)} \]

Note: This is about 15.373 meters. Do not expect to evaluate this exactly!

7) The graph of \( y = x^4 + 3x^2 + 2 \) from \((-1, 6)\) to \((2, 30)\) is revolved about the \( x \)-axis. Set up the integral for the area of the resulting surface. Your final variable of integration must be \( x \) or \( y \), as used in this problem. **You do not have to sketch a graph. Do not evaluate.** (10 points)

Let \( f(x) = x^4 + 3x^2 + 2 \). Then, \( f \) is nonnegative and continuous on the \( x \)-interval \([-1, 2]\).

\( f''(x) = 4x^3 + 6x \), so \( f' \) is also continuous there.

For the desired surface area, we integrate circumferences with respect to arc length.

Surface Area, \( S = \int_{x=-1}^{x=2} 2\pi f(x) \, ds \)
\[ = \int_{-1}^{2} 2\pi f(x) \sqrt{1 + [f'(x)]^2} \, dx \]
\[ = \int_{-1}^{2} 2\pi \left( x^4 + 3x^2 + 2 \right) \sqrt{1 + [4x^3 + 6x]^2} \, dx, \text{ or} \]
\[ \int_{-1}^{2} 2\pi \left( x^4 + 3x^2 + 2 \right) \sqrt{16x^6 + 48x^4 + 36x^2 + 1} \, dx \text{ (in m}^2 \text{)} \]

Note: This is about 2924.5 m\(^2\). Do not expect to evaluate this exactly!