

QUIZ ON CHAPTER 6 - SOLUTIONS

APPLICATIONS OF INTEGRALS; MATH 150 – FALL 2016 – KUNIYUKI 105 POINTS TOTAL, BUT 100 POINTS = 100%

Note: The functions here are continuous on the intervals of interest. This guarantees integrability and the validity of the Test Value Method to determine relative positions on an interval (that is, which graph is on top vs. on bottom, or on the right vs. on the left). The FTC applies.

- 1) Find the area of the region R bounded by the graphs of $x = y^2 - 4$ and $x + y = 2$. **Sketch and shade in the region R , and identify any intersection points and intercepts. Evaluate your integral completely, and write your answer as an exact fraction or an exact mixed number in simplest form, together with appropriate units. Show all work. (25 points)**

- The given equation $x = y^2 - 4$ is solved for x in terms of y . We hope to do a “ dy scan.”
- Find the y -coordinates of the intersection points of the corresponding graphs by solving the following system for y ; they will be the limits of integration.

$$\begin{cases} x = y^2 - 4 \\ x + y = 2 \end{cases} \quad \left(\text{for } dy \text{ scan, solve for } x \text{ in terms of } y \right) \Leftrightarrow \begin{cases} x = y^2 - 4 & \text{(parabola opening right; leading coefficient on right is positive)} \\ x = 2 - y & \text{(line)} \end{cases}$$

We will equate the expressions for x :

$$\begin{aligned} y^2 - 4 &= 2 - y \\ y^2 + y - 6 &= 0 \\ (y + 3)(y - 2) &= 0 \\ y + 3 = 0 &\quad \text{or} \quad y - 2 = 0 \\ y = -3 &\quad \quad y = 2 \end{aligned}$$

Our limits of integration will apparently be $a = -3$ and $b = 2$.

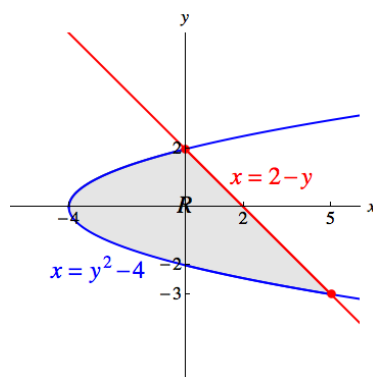
- We will use the equation $x = 2 - y$ to determine the x -coordinates of the intersection points. (We could also use the equation $x = y^2 - 4$.)

$$\begin{array}{lclclcl} y = -3 & \Rightarrow & x = 2 - (-3) & \Rightarrow & x = 5 & \Rightarrow & \text{Point } (5, -3) \\ y = 2 & \Rightarrow & x = 2 - (2) & \Rightarrow & x = 0 & \Rightarrow & \text{Point } (0, 2) \end{array}$$

The point $(0, 2)$ is the sole y -intercept of the line. It is also one of two y -intercepts for the parabola; the other is at $(0, -2)$, since 2 and -2 are real zeros of $y^2 - 4$.

- Which forms the **right** boundary of R : the parabola or the line? Which forms the **left**?
 - The **parabola** opens to the right, so it forms the **left** boundary of R .
 - Test a y -value in the interval $(-3, 2)$, say $y = 0$; we then identify x -intercepts.
Graph of **parabola**: $x = (0)^2 - 4 \Rightarrow x = -4 \quad [\Rightarrow \text{ } x\text{-int. is } (-4, 0)]$.
Graph of **line**: $x = 2 - (0) \Rightarrow x = 2 \quad [\Rightarrow \text{ } x\text{-int. is } (2, 0)]$.

$2 > -4$, so the **line** forms the **right** boundary of R .

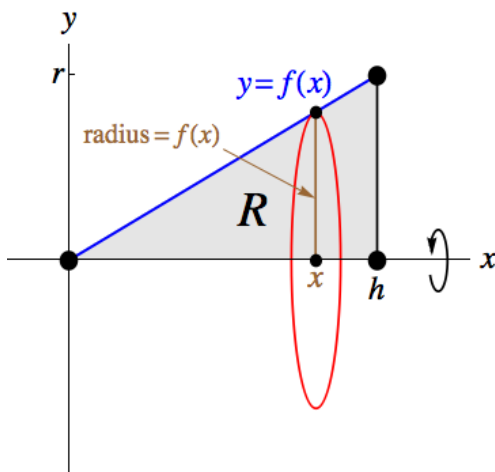


- We will do a “ dy scan,” because the region R is well suited to that, and the given equations are easily solved for x in terms of y .
- We’re not revolving R about any axis, so the fact that the x - and y -axes pass through the interior of R does not pose a problem.
- A , the area of R , is given by:

$$\begin{aligned}
 A &= \int_{-3}^1 \left[\underbrace{(2-y)}_{x_{\text{right}}(y)} - \underbrace{(y^2-4)}_{x_{\text{left}}(y)} \right] dy = \int_{-3}^1 [2-y-y^2+4] dy = \int_{-3}^1 [-y^2-y+6] dy \\
 &= \left[-\frac{y^3}{3} - \frac{y^2}{2} + 6y \right]_{-3}^1 = \left[-\frac{(1)^3}{3} - \frac{(1)^2}{2} + 6(1) \right] - \left[-\frac{(-3)^3}{3} - \frac{(-3)^2}{2} + 6(-3) \right] \\
 &= \left[-\frac{1}{3} - \frac{1}{2} + 6 \right] - \left[9 - \frac{9}{2} - 18 \right] = \left[-\frac{1}{3} + \frac{11}{2} \right] - \left[-\frac{9}{2} - 9 \right] = -\frac{1}{3} + \frac{11}{2} + \frac{9}{2} + 9 \\
 &= \frac{9}{2} - \frac{1}{3} + 19 = \frac{27}{6} - \frac{2}{6} + 19 = \frac{25}{6} + 19 = 1\frac{5}{6} + 19 = \boxed{20\frac{5}{6} \text{ m}^2, \text{ or } \frac{125}{6} \text{ m}^2}
 \end{aligned}$$

2) Using the Disk Method, as we have discussed in class, find the volume of a right circular cone of altitude h and base radius r . Show all work. (20 points)

- Such a cone is obtained by revolving the triangular region R about the x -axis.
- We are revolving R about a horizontal axis, so the Disk Method requires a “ dx scan.”
- Observe that the axis of revolution does not pass through the interior of R .



- Find the function f such that $y = f(x)$ models the slanted line segment.

$$y = mx + b$$

$$y = \left(\frac{\text{rise}}{\text{run}} \right) x + 0$$

$$y = \frac{r}{h} x \quad (0 \leq x \leq h)$$

- We will apply the Disk Method to find the volume (V) of the cone.

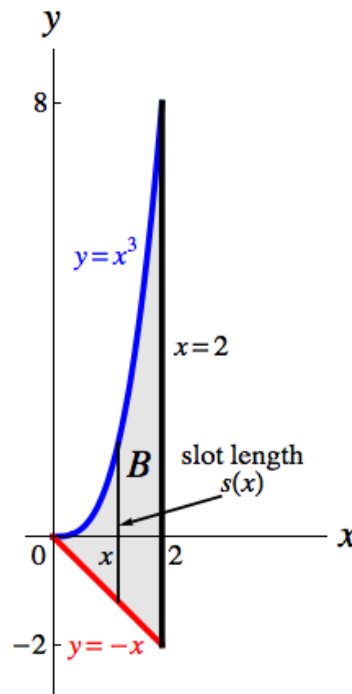
$$\begin{aligned} V &= \int_0^h \pi (\text{radius})^2 dx = \int_0^h \pi \left(\frac{r}{h} x \right)^2 dx = \pi \int_0^h \frac{r^2}{h^2} x^2 dx \quad \left(\text{Key: } \frac{r^2}{h^2} \text{ is a constant.} \right) \\ &= \frac{\pi r^2}{h^2} \int_0^h x^2 dx = \frac{\pi r^2}{h^2} \left[\frac{x^3}{3} \right]_0^h = \frac{\pi r^2}{h^2} \left(\left[\frac{(h)^3}{3} \right] - [0] \right) = \frac{\pi r^2}{\cancel{h^2}^{(1)}} \left(\frac{\cancel{h^3}^{(h)}}{3} \right) = \boxed{\frac{1}{3} \pi r^2 h \text{ (in m}^3\text{)}} \end{aligned}$$

- 3) Let B be the region bounded by the graphs of $y = x^3$, $y = -x$, and $x = 2$.

Sketch and shade in the region B . Find the volume of the solid that has B as its base if every cross section by a plane perpendicular to the x -axis is a square.

Evaluate your integral completely, and give appropriate units. You may **round off** your final answer to four significant digits. Show all work. (14 points)

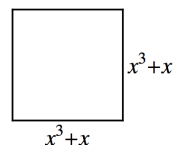
- The cross-sections are perpendicular to the x -axis, so we will integrate with respect to x (“ dx scan”). The first two given equations are already solved for y in terms of x .



- Fix a generic x -value in the interval $[0, 2]$. The cross-section at x is a square (which degenerates to a point at $x = 0$) whose side length is:

$$\begin{aligned} s(x) &= y_{\text{top}}(x) - y_{\text{bottom}}(x) \\ &= x^3 - (-x) \\ &= x^3 + x \end{aligned}$$

- The area of the square is: $A(x) = [s(x)]^2 = (x^3 + x)^2$.



- V , the volume of the solid, is given by:

$$\begin{aligned}
 V &= \int_0^2 A(x) dx = \int_0^2 (x^3 + x)^2 dx = \int_0^2 (x^6 + 2x^4 + x^2) dx = \left[\frac{x^7}{7} + \frac{2x^5}{5} + \frac{x^3}{3} \right]_0^2 \\
 &= \left[\frac{(2)^7}{7} + \frac{2(2)^5}{5} + \frac{(2)^3}{3} \right] - [0] = \frac{128}{7} + \frac{64}{5} + \frac{8}{3} = \frac{1920}{105} + \frac{1344}{105} + \frac{280}{105} \\
 &= \boxed{\frac{3544}{105} \text{ m}^3, \text{ or } 33 \frac{79}{105} \text{ m}^3 \approx 33.75 \text{ m}^3}
 \end{aligned}$$

- 4) The region R is bounded by the graphs of $y = 0$, $y = x$, and $5x + 2y = 10$.

Sketch and shade in the region R , and identify any intersection points and intercepts. Set up the integral for the volume of the solid generated if R is revolved about the y -axis. **Do not evaluate.** Use the Washer Method.
(15 points)

- We are revolving R about a vertical axis (the y -axis), so we must do a “ dy scan” to use the Washer Method. Solve for x in terms of y in the last two given equations.

$$\begin{aligned}
 y &= x & 5x + 2y &= 10 \\
 x &= y & 5x &= 10 - 2y \\
 & & x &= 2 - \frac{2}{5}y
 \end{aligned}$$

- Find the intersection point shared by the corresponding lines:

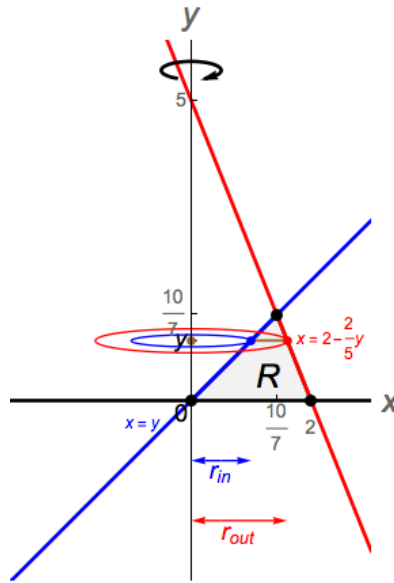
Find the y -coordinate by solving the system $\begin{cases} x = y \\ x = 2 - \frac{2}{5}y \end{cases}$ for y .

We will equate the expressions for x :

$$\begin{aligned}
 y &= 2 - \frac{2}{5}y \\
 \frac{7}{5}y &= 2 \\
 y &= \frac{10}{7}
 \end{aligned}$$

Then, $x = y \Rightarrow x = \frac{10}{7}$. The intersection point is: $\boxed{\left(\frac{10}{7}, \frac{10}{7} \right)}$.

- The graph of $x = 2 - \frac{2}{5}y$ is the **red** line that forms the right boundary of R .
 - It forms the outer boundary of R relative to the axis of revolution.
 - Looking at the original equation, $5x + 2y = 10$, observe that the x -intercept is at $(2, 0)$, and the y -intercept is at $(0, 5)$.
- The graph of $x = y$ is the **blue** line that forms the left boundary of R .
 - It forms the inner boundary of R relative to the axis of revolution.
 - The point $(0, 0)$ is both the sole x -intercept and the sole y -intercept.



- The region R is well-suited to a “ dy scan.”
- The y -axis (the axis of revolution) does not pass through the interior of R .

Fix a generic y -value in the interval $\left[0, \frac{10}{7}\right]$.

Find the radii $r_{out}(y)$ and $r_{in}(y)$ of the corresponding washer:

$$\begin{aligned}
 r_{out}(y) &= x_{right}(y) - x_{left}(y) & r_{in}(y) &= x_{right}(y) - x_{left}(y) \\
 &= \left(2 - \frac{2}{5}y\right) - (0) & &= (y) - (0) \\
 &= 2 - \frac{2}{5}y & &= y
 \end{aligned}$$

V , the volume of the solid, is given by:

$$\begin{aligned}
 V &= \int_0^{10/7} \left(\pi [r_{out}(y)]^2 - \pi [r_{in}(y)]^2 \right) dy \\
 &= \boxed{ \int_0^{10/7} \left[\pi \left(2 - \frac{2}{5}y \right)^2 - \pi (y)^2 \right] dy, \quad \text{or} \quad \pi \int_0^{10/7} \left[\left(2 - \frac{2}{5}y \right)^2 - (y)^2 \right] dy, \quad \text{or} } \\
 &\quad \pi \int_0^{10/7} \left[4 - \frac{8}{5}y - \frac{21}{25}y^2 \right] dy \quad (\text{in m}^3)
 \end{aligned}$$

Note 1: If you did not know that the graph of $x = 2 - \frac{2}{5}y$ lay to the right of the graph of $x = y$, then test (say) $y = 1$ to see that $2 - \frac{2}{5}y \geq y$ on the y -interval $\left[0, \frac{10}{7}\right]$.

Note 2: The volume is: $\frac{160\pi}{49} \text{ m}^3 \approx 10.258 \text{ m}^3$. Check this for yourself!

5) The region R lies only in Quadrant I and is bounded by the graphs of $y = x^3$, $y = 3x^3$, and $x = 1$. **Sketch and shade in the region R . Set up the integral** for the volume of the solid generated if R is revolved about the line $x = 2$. **Do not evaluate.** Use the Cylindrical Shell (Cylinder) Method. (15 points)

- We are revolving R about a vertical axis ($x = 2$), so we must do a “ dx scan” to use the Cylindrical Shell (Cylinder) Method.
- We already have the first two given equations solved for y in terms of x .
- The graphs of $y = x^3$ and $y = 3x^3$ intersect at $(0, 0)$ but at no other point:

Consider the system: $\begin{cases} y = x^3 \\ y = 3x^3 \end{cases}$. Equate the expressions for y and solve for x :

$$x^3 = 3x^3$$

$$0 = 2x^3$$

$$x = 0$$

$$x = 0 \Rightarrow y = 0 \Rightarrow (0, 0) \text{ is the [only] intersection point.}$$

- Which graph forms the **top** boundary of R on the x -interval $[0, 1]$: $y = x^3$ or $y = 3x^3$?

The related functions are continuous on $[0, 1]$, with graphs only intersecting at $(0, 0)$. Test any x -value in the interval $(0, 1)$. If we test $x = \frac{1}{2}$, for example:

$$y = x^3 \Rightarrow y = \left(\frac{1}{2}\right)^3 = \frac{1}{8} \quad (\text{bottom})$$

$$y = 3x^3 \Rightarrow y = 3\left(\frac{1}{2}\right)^3 = \frac{3}{8} \quad (\text{top})$$

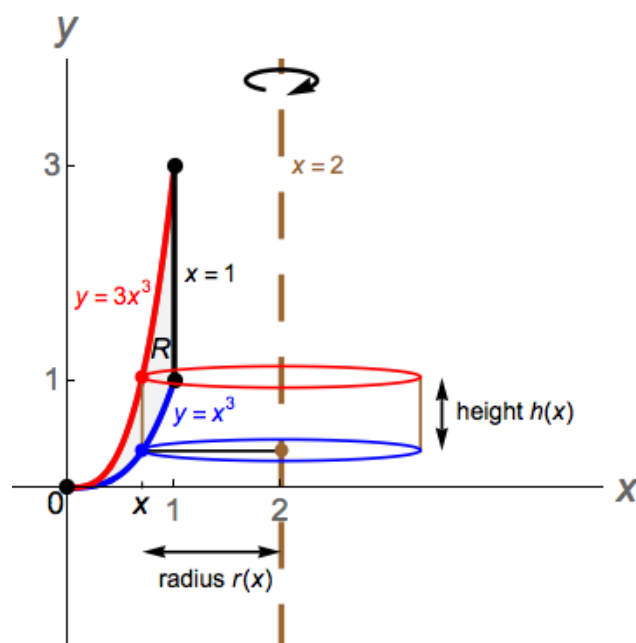
$\frac{3}{8} > \frac{1}{8}$, so the graph of **$y = 3x^3$** is on **top**; it forms the top boundary of R .

- The “corners” of R are at $(0, 0)$, $(1, 1)$, and $(1, 3)$. The last two lie on the line $x = 1$:

$$\text{On the graph of } y = x^3: x = 1 \Rightarrow y = (1)^3 = 1 \Rightarrow \text{Point}(1, 1)$$

$$\text{On the graph of } y = 3x^3: x = 1 \Rightarrow y = 3(1)^3 = 3 \Rightarrow \text{Point}(1, 3)$$

- Observe that the axis of revolution ($x = 2$) does not pass through the interior of R .



Fix a generic x -value in the interval $[0, 1]$.

Find the radius $r(x)$ and the height $h(x)$ of the corresponding cylinder:

$$\begin{aligned} \text{radius } r(x) &= x_{\text{right}}(x) - x_{\text{left}}(x) & \text{height } h(x) &= y_{\text{top}}(x) - y_{\text{bottom}}(x) \\ &= (2) - (x) & &= (3x^3) - (x^3) \\ &= 2 - x & &= 2x^3 \end{aligned}$$

V , the volume of the solid, is given by:

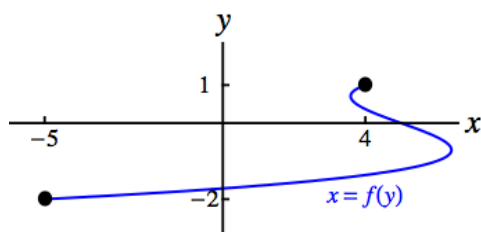
$$\begin{aligned} V &= \int_0^1 2\pi [\text{radius } r(x)] [\text{height } h(x)] dx \\ &= \boxed{\int_0^1 2\pi (2-x)(2x^3) dx, \text{ or } 4\pi \int_0^1 (2x^3 - x^4) dx \quad (\text{in m}^3)} \end{aligned}$$

Note: The volume is $\frac{6\pi}{5} \text{ m}^3 \approx 3.770 \text{ m}^3$. Check this for yourself!

- 6) **Set up the integral** for the arc length of the graph of $x = 2y^3 - 3y + 5$ from $(-5, -2)$ to $(4, 1)$. Your final variable of integration must be x or y , as used in this problem. **You do not have to sketch a graph. Do not evaluate.** (6 points)

Let $f(y) = 2y^3 - 3y + 5$. Then, f is continuous on the y -interval $[-2, 1]$.

$f'(y) = 6y^2 - 3$, so f' is also continuous there.



(The x - and y -axes are scaled differently.)

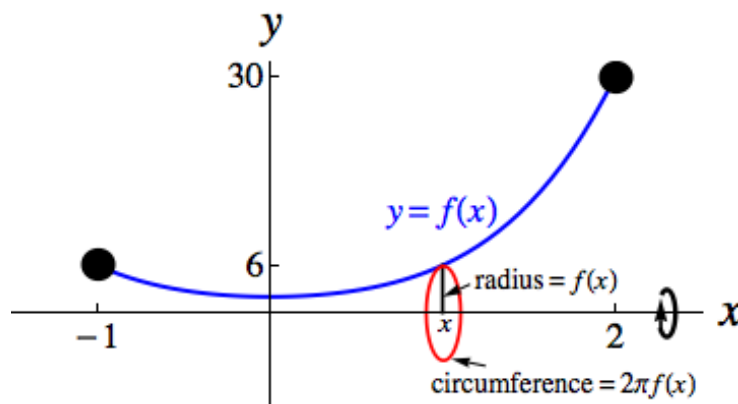
$$\begin{aligned}
 \text{Arc length, } L &= \int_{y=-2}^{y=1} ds \\
 &= \int_{-2}^1 \sqrt{1 + [f'(y)]^2} dy \\
 &= \int_{-2}^1 \sqrt{1 + [6y^2 - 3]^2} dy, \text{ or} \\
 &= \int_{-2}^1 \sqrt{36y^4 - 36y^2 + 10} dy \quad (\text{in meters})
 \end{aligned}$$

Note: This is about 15.373 meters. Do not expect to evaluate this exactly!

- 7) The graph of $y = x^4 + 3x^2 + 2$ from $(-1, 6)$ to $(2, 30)$ is revolved about the x -axis. **Set up the integral** for the area of the resulting surface. Your final variable of integration must be x or y , as used in this problem. **You do not have to sketch a graph. Do not evaluate.** (10 points)

Let $f(x) = x^4 + 3x^2 + 2$. Then, f is nonnegative and continuous on the x -interval $[-1, 2]$.

$f'(x) = 4x^3 + 6x$, so f' is also continuous there.



(The x - and y -axes are scaled differently.)

For the desired surface area, we integrate circumferences with respect to arc length.

$$\begin{aligned}
 \text{Surface Area, } S &= \int_{x=-1}^{x=2} 2\pi f(x) ds \\
 &= \int_{-1}^2 2\pi f(x) \sqrt{1 + [f'(x)]^2} dx \\
 &= \int_{-1}^2 2\pi (x^4 + 3x^2 + 2) \sqrt{1 + [4x^3 + 6x]^2} dx, \text{ or} \\
 &= \int_{-1}^2 2\pi (x^4 + 3x^2 + 2) \sqrt{16x^6 + 48x^4 + 36x^2 + 1} dx \quad (\text{in m}^2)
 \end{aligned}$$

Note: This is about 2924.5 m^2 . Do not expect to evaluate this exactly!