# QUIZ ON CHAPTER 6 - SOLUTIONS 

APPLICATIONS OF INTEGRALS; MATH 150 - SPRING 2017 - KUNIYUKI 105 POINTS TOTAL, BUT 100 POINTS = 100\%
Note: The functions here are continuous on the intervals of interest. This guarantees integrability and the validity of the Test Value Method to determine relative positions on an interval (that is, which graph is on top vs. on bottom, or on the right vs. on the left). The FTC applies.

1) Find the area of the region $R$ bounded by the graphs of $y=2 x^{2}+3 x-4$ and $y=x^{2}+5 x+4$. Identify any intersection points. You do not have to sketch the region or find intercepts. Evaluate your integral completely, and give your final answer with appropriate units. Show all work. (18 points)

- Both given equations are solved for $y$ in terms of $x$, so we hope to do a " $d x$ scan."
- Find the $x$-coordinates of the intersection points of the corresponding graphs by solving the following system for $x$; they will be the limits of integration.

$$
\left\{\begin{array}{l}
y=2 x^{2}+3 x-4 \\
y=x^{2}+5 x+4
\end{array} \Rightarrow\right.
$$

We will equate the expressions for $y$ :

$$
\begin{array}{rlrl}
2 x^{2}+3 x-4 & =x^{2}+5 x+4 \\
x^{2}-2 x-8 & =0 \\
(x-4)(x+2) & =0 \\
x-4=0 & \text { or } \quad x+2 & =0 \\
x=4 \quad x & =-2
\end{array}
$$

Our limits of integration will apparently be $a=-2$ and $b=4$.

- We will use the equation $y=x^{2}+5 x+4$ to determine the $y$-coordinates of the intersection points. (We could also use the equation $y=2 x^{2}+3 x-4$.)

$$
\begin{aligned}
& x=-2 \Rightarrow y=(-2)^{2}+5(-2)+4 \Rightarrow y=-2 \Rightarrow \operatorname{Point}(-2,-2) \\
& x=4 \Rightarrow y=(4)^{2}+5(4)+4 \Rightarrow y=40 \Rightarrow \operatorname{Point}(4,40)
\end{aligned}
$$

- The graphs of $y=x^{2}+5 x+4$ and $y=2 x^{2}+3 x-4$ are upward-opening parabolas.
- Which parabola forms the top boundary of $R$, and which forms the bottom boundary?

We can test an $x$ value in the interval $(-2,4)$, say $x=0$.
(Then, we're identifying $y$-intercepts, also.)

$$
\begin{aligned}
& \text { Graph of } y=2 x^{2}+3 x-4: y=2(0)^{2}+3(0)-4 \Rightarrow y=-4[\Rightarrow y \text {-int. is }(0,-4)] \\
& \text { Graph of } y=x^{2}+5 x+4: \quad y=(0)^{2}+5(0)+4 \Rightarrow y=4 \quad[\Rightarrow y \text {-int. is }(0,4)]
\end{aligned}
$$

$4>-4$, so this graph forms the top boundary of $R$.

- Here is a sketch of $R$; the $x$ - and $y$-axes are scaled differently:

- The region $R$ is well-suited to a " $d x$ scan."
- We're not revolving $R$ about any axis, so the fact that the $x$-axis and the $y$-axis pass through the interior of $R$ does not pose a problem.
- $A$, the area of $R$, is given by:

$$
\begin{aligned}
& A=\int_{-2}^{4}[\underbrace{\left(x^{2}+5 x+4\right)}_{y_{\text {top }}(x)}-\underbrace{\left(2 x^{2}+3 x-4\right)}_{y_{\text {bottom }}(x)}] d x=\int_{-2}^{4}\left[x^{2}+5 x+4-2 x^{2}-3 x+4\right] d x \\
& =\int_{-2}^{4}\left[-x^{2}+2 x+8\right] d x=\left[-\frac{x^{3}}{3}+x^{2}+8 x\right]_{-2}^{4} \\
& =\left[-\frac{(4)^{3}}{3}+(4)^{2}+8(4)\right]-\left[-\frac{(-2)^{3}}{3}+(-2)^{2}+8(-2)\right] \\
& =\left[-\frac{64}{3}+16+32\right]-\left[\frac{8}{3}+4-16\right]=\left[-\frac{64}{3}+48\right]-\left[\frac{8}{3}-12\right]=-\frac{64}{3}+48-\frac{8}{3}+12 \\
& =60-\frac{72}{3}=60-24=36 \mathrm{~m}^{2}
\end{aligned}
$$

2) Using the Disk Method, as we have discussed in class, find the volume of a sphere of radius $r$. Show all work. ( 20 points)

The desired volume is twice the volume of the hemisphere obtained by revolving the quarter-circular region $R$ below about the $x$-axis.

- We are revolving $R$ about a horizontal axis, so the Disk Method requires a " $d x$ scan."
- Observe that the axis of revolution does not pass through the interior of $R$.


Model the quarter-circle by finding the corresponding function $f$ such that $y=f(x)$.

$$
\begin{aligned}
x^{2}+y^{2} & =r^{2} \\
y^{2} & =r^{2}-x^{2} \\
y & = \pm \sqrt{r^{2}-x^{2}} \\
\text { Take } y & =\sqrt{r^{2}-x^{2}}, 0 \leq x \leq r \text { (giving the indicated quarter-circle). }
\end{aligned}
$$

We will apply the Disk Method to find the volume $(V)$ of the sphere.

$$
\begin{aligned}
& V=2(\text { Volume of hemisphere })=2 \int_{0}^{r} \pi(\text { radius })^{2} d x=2 \int_{0}^{r} \pi\left(\sqrt{r^{2}-x^{2}}\right)^{2} d x \\
& =2 \pi \int_{0}^{r}\left(r^{2}-x^{2}\right) d x=2 \pi\left[r^{2} x-\frac{x^{3}}{3}\right]_{0}^{r} \quad\left(\text { Remember: } r^{2} \text { is a constant. }\right) \\
& =2 \pi\left(\left[r^{2}(r)-\frac{(r)^{3}}{3}\right]-[0]=2 \pi\left(r^{3}-\frac{1}{3} r^{3}\right)=2 \pi\left(\frac{2}{3} r^{3}\right)=\frac{4}{3} \pi r^{3}\left(\text { in m}^{3}\right)\right.
\end{aligned}
$$

3) $B$ is the region in Quadrant I of the $x y$-plane that is bounded by the $x$-axis, the $y$-axis, and the graph of $y=9-x^{2}$. Sketch and shade in the region $\boldsymbol{B}$. Find the volume of the solid that has $B$ as its base if every cross-section by a plane perpendicular to the $x$-axis is a semicircular region with diameter in the $x y$-plane. Evaluate your integral completely, and write your answer as an exact fraction in simplest form, together with appropriate units. Show all work. (18 points)

- The cross-sections are perpendicular to the $x$-axis, so we will integrate with respect to $x$ (" $d x$ scan"). The given equation is already solved for $y$ in terms of $x$.
- The graph of $y=9-x^{2}$ intersects the $x$-axis $(y=0)$ along the boundary of Quadrant I at the point $(3,0)$, where $x=3$. This is because:

$$
\left\{\begin{aligned}
y=9-x^{2} \\
y=0
\end{aligned} \Rightarrow \begin{array}{rl}
9-x^{2} & =0 \\
x^{2} & =9 \\
x & = \pm 3 \\
\text { Take } x & =3 .
\end{array}\right.
$$

- Our limits of integration will apparently be $a=0$ (due to the $y$-axis, $x=0$ ) and $b=3$.
- Here is a sketch of $B$ :

- Fix a generic $x$-value in the interval $[0,3]$. The cross-section at $x$ is a semicircular region (which degenerates to a point at $x=3$ ) of diameter $9-x^{2}$ and radius $\frac{1}{2}\left(9-x^{2}\right)$.
- The area of this cross-section is:

$$
A(x)=\frac{1}{2} \pi[\text { radius }]^{2}=\frac{1}{2} \pi\left[\frac{1}{2}\left(9-x^{2}\right)\right]^{2}=\frac{1}{2} \pi\left[\frac{1}{4}\left(9-x^{2}\right)^{2}\right]=\frac{\pi}{8}\left(9-x^{2}\right)^{2}
$$

Note: At the degenerate case of $x=3$, we correctly obtain zero area.

- $V$, the volume of the solid, is given by:

$$
\begin{aligned}
& V=\int_{0}^{3} A(x) d x=\int_{0}^{3} \frac{\pi}{8}\left(9-x^{2}\right)^{2} d x=\frac{\pi}{8} \int_{0}^{3}\left(81-18 x^{2}+x^{4}\right) d x \\
& =\frac{\pi}{8}\left[81 x-18\left(\frac{x^{3}}{3}\right)+\frac{x^{5}}{5}\right]_{0}^{3}=\frac{\pi}{8}\left[81 x-6 x^{3}+\frac{x^{5}}{5}\right]_{0}^{3} \\
& =\frac{\pi}{8}\left[\left[81(3)-6(3)^{3}+\frac{(3)^{5}}{5}\right]-[0]\right)=\frac{\pi}{8}\left[243-162+\frac{243}{5}\right] \\
& \left.=\frac{\pi}{8}\left[\frac{1215-810+243}{5}\right]=\frac{\pi}{8}\left[\frac{648}{5}\right]=\frac{81 \pi}{5} \mathrm{~m}^{3}\right] \approx 50.894 \mathrm{~m}^{3}
\end{aligned}
$$

4) The region $R$ is bounded by the graphs of $x=y^{2}$ and $x+y=6$. Sketch and shade in the region $R$, and identify any intersection points and intercepts.
Set up the integral for the volume of the solid generated if $R$ is revolved about the $y$-axis. Do not evaluate. Use the Washer Method. (18 points)

- We are revolving $R$ about a vertical axis (the $y$-axis), so do a " $d y$ scan" to use the Washer Method. Solve the second given equation for $x$ in terms of $y$, and find the $y$-coordinates of the intersection points; they will be the limits of integration.

$$
\left\{\begin{array} { c } 
{ x = y ^ { 2 } } \\
{ x + y = 6 }
\end{array} \Rightarrow \left\{\begin{array}{l}
x=y^{2} \\
x=6-y
\end{array}\right.\right.
$$

We will equate the expressions for $x$ :

$$
\begin{gathered}
y^{2}=6-y \\
y^{2}+y-6=0 \\
(y+3)(y-2)=0 \\
\left.y+3=0 \quad{ }^{2} \quad \begin{array}{rl}
y-2 & =0 \\
y=-3 & \text { or } \quad y
\end{array}\right)=2
\end{gathered}
$$

Our limits of integration will apparently be $a=-3$ and $b=2$.
Find the intersection points:

$$
\begin{aligned}
& y=-3 \Rightarrow x=(-3)^{2}=9 \Rightarrow \text { Intersection point }(9,-3) . \\
& y=2 \Rightarrow x=(2)^{2}=4 \Rightarrow \text { Intersection point }(4,2) .
\end{aligned}
$$

The graph of $x=y^{2}$ is the rightward-opening parabola that forms the left boundary of $R$.

- More importantly, it forms the inner boundary of $R$ relative to the axis of revolution.
- The point $(0,0)$ is both the sole $x$-intercept and the sole $y$-intercept.

The graph of $x=6-y$ is the line that forms the right boundary of $R$.

- More importantly, it forms the outer boundary of $R$ relative to the axis of revolution.
- The $x$-intercept is at $(6,0)$, and the $y$-intercept is at $(0,6)$.

We could test, say, $y=0$ to see that $6-y \geq y^{2}$ on the $y$-interval $[-3,2]$.


- The region $R$ is well-suited to a " $d y$ scan."
- The $y$-axis (the axis of revolution) does not pass through the interior of $R$. Fix a generic $y$-value in the interval $[-3,2]$.
Find the radii $r_{\text {out }}(y)$ and $r_{\text {in }}(y)$ of the corresponding washer:

$$
\begin{aligned}
r_{\text {out }}(y) & =x_{\text {right }}(y)-x_{\text {left }}(y) & r_{\text {in }}(y) & =x_{\text {right }}(y)-x_{\text {left }}(y) \\
& =(6-y)-(0) & & =\left(y^{2}\right)-(0) \\
& =6-y & & =y^{2}
\end{aligned}
$$

$V$, the volume of the solid, is given by:

$$
\begin{aligned}
V & =\int_{-3}^{2}\left(\pi\left[r_{\text {out }}(y)\right]^{2}-\pi\left[r_{\text {in }}(y)\right]^{2}\right) d y \\
& =\begin{array}{l}
\int_{-3}^{2}\left[\pi(6-y)^{2}-\pi\left(y^{2}\right)^{2}\right] d y, \text { or } \pi \int_{-3}^{2}\left[(6-y)^{2}-y^{4}\right] d y, \quad \text { or } \\
\pi \int_{-3}^{2}\left[36-12 y+y^{2}-y^{4}\right] d y\left(\text { in m }^{3}\right)
\end{array}
\end{aligned}
$$

Note: The volume is: $\frac{500 \pi}{3} \approx 523.60 \mathrm{~m}^{3}$. Check this for yourself!
5) The region $R$ is bounded by the $x$-axis and the graphs of $y=x^{3}$ and $x=2$. Sketch and shade in the region $R$, and identify any intersection points and intercepts. Set up the integral for the volume of the solid generated if $R$ is revolved about the line $y=-2$. Do not evaluate. Use the Cylindrical Shell (Cylinder) Method. (15 points)

- We are revolving $R$ about a horizontal axis (parallel to the $x$-axis), so do a
" $d y$ scan" to use the Cylindrical Shell (Cylinder) Method.
- Solve the equation $y=x^{3}$ for $x$ in terms of $y$.

$$
y=x^{3} \Leftrightarrow x=\sqrt[3]{y}
$$

- By solving the system: $\left\{\begin{array}{l}y=x^{3} \\ x=2\end{array} \Rightarrow\left(y=(2)^{3}=8\right)\right.$,
we see that the graph of $y=x^{3}$ intersects the line $x=2$ at the point $(2,8)$, where $y=8$.
- By solving the system: $\left\{\begin{array}{ll}y=x^{3}, & \text { or } x=\sqrt[3]{y} \\ y=0 & (\text { the } x \text {-axis })\end{array} \Rightarrow(x=\sqrt[3]{0}=0)\right.$,
we see that the graph of $y=x^{3}$ intersects the $x$-axis at the point $(0,0)$, where $y=0$.
- Our limits of integration will apparently be $a=0$ (corresponding to the $x$-axis, $y=0$ ) and $b=8$.

- Observe that the axis of revolution $(y=-2)$ does not pass through the interior of $R$.
- (The $x$ - and $y$-axes are scaled differently.)

Fix a generic $y$-value in the interval $[0,8]$.
Find the radius $r(y)$ and the height $h(y)$ of the corresponding cylinder:

$$
\text { radius } \begin{aligned}
r(y) & =y_{\text {top }}(y)-y_{\text {botoom }}(y) & \text { height } \begin{aligned}
h(y) & =x_{\text {right }}(y)-x_{\text {left }}(y) \\
& =y-(-2) \\
& =2-\sqrt[3]{y} \\
& =y+2
\end{aligned} &
\end{aligned}
$$

$V$, the volume of the solid, is given by:

$$
\begin{aligned}
V & =\int_{0}^{8} 2 \pi[\text { radius } r(y)][\text { height } h(y)] d y \\
& =\begin{array}{l}
\int_{0}^{8} 2 \pi(y+2)(2-\sqrt[3]{y}) d y, \text { or } \\
2 \pi \int_{0}^{8}\left(2 y-y^{4 / 3}+4-2 y^{1 / 3}\right) d y\left(\text { in }^{3}\right)
\end{array}
\end{aligned}
$$

Note: The volume is $\frac{240 \pi}{7} \approx 107.71 \mathrm{~m}^{3}$ cubic meters. Check this for yourself!
6) Set up the integral for the arc length of the graph of $x=y^{2}+y$ from $(2,1)$ to $(20,4)$. Your final variable of integration must be $x$ or $y$, as used in this problem. Do not evaluate. You do not have to sketch a graph. (6 points)

Let $f(y)=y^{2}+y$. Then, $f$ is continuous on the $y$-interval $[1,4]$.
$f^{\prime}(y)=2 y+1$, so $f^{\prime}$ is also continuous there.


Arc length, $L=\int_{y=1}^{y=4} d s=\int_{y=1}^{y=4} \sqrt{1+\left[f^{\prime}(y)\right]^{2}} d y$
$=\sqrt{\int_{1}^{4} \sqrt{1+[2 y+1]^{2}} d y \text {, or } \int_{1}^{4} \sqrt{4 y^{2}+4 y+2} d y \quad(\text { in meters })}$
Note: This is about 18.27 meters. You would not be expected to evaluate this exactly!
7) The graph of $y=\sqrt[3]{x}$ from $(8,2)$ to $(27,3)$ is revolved about the $x$-axis.

Set up the integral for the area of the resulting surface. Your final variable of integration must be $x$ or $y$, as used in this problem. Do not evaluate. You do not have to sketch a graph. (10 points)

Let $f(x)=\sqrt[3]{x}=x^{1 / 3}$. Then, $f$ is nonnegative and continuous on the $x$-interval $[8,27]$.
$f^{\prime}(x)=\frac{1}{3} x^{-2 / 3}=\frac{1}{3 x^{2 / 3}}$, so $f^{\prime}$ is also continuous there.
(2)

For the desired surface area, we integrate circumferences with respect to arc length.

$$
\begin{aligned}
& \text { Surface Area, } S=\int_{x=8}^{x=27} 2 \pi f(x) d s=\int_{x=8}^{x=27} 2 \pi f(x) \sqrt{1+\left[f^{\prime}(x)\right]^{2}} d x \\
& =\int_{8}^{27} 2 \pi(\sqrt[3]{x}) \sqrt{1+\left[\frac{1}{3} x^{-2 / 3}\right]^{2}} d x, \text { or } \int_{8}^{27} 2 \pi(\sqrt[3]{x}) \sqrt{1+\left[\frac{1}{3 x^{2 / 3}}\right]^{2}} d x, \\
& \text { or } \int_{8}^{27} 2 \pi(\sqrt[3]{x}) \sqrt{1+\frac{1}{9 x^{4 / 3}}} d x, \text { or } \int_{8}^{27} 2 \pi(\sqrt[3]{x}) \sqrt{\frac{9 x^{4 / 3}+1}{9 x^{4 / 3}}} d x, \\
& \text { or } \int_{8}^{27} 2 \pi(\sqrt[3]{x}) \frac{\sqrt{9 x^{4 / 3}+1}}{3 x^{2 / 3}} d x, \text { or } 2 \pi \int_{8}^{27} \frac{\sqrt{9 x^{4 / 3}+1}}{3 x^{1 / 3}} d x\left(\text { in m }^{2}\right)
\end{aligned}
$$

Note: This is about 306.7 square meters. You would not be expected to evaluate this exactly!

