QUIZ ON CHAPTER 7 - SOLUTIONS

LOG AND EXPONENTIAL FUNCTIONS; MATH 150 – SPRING 2017 – KUNIYUKI
105 POINTS TOTAL, BUT 100 POINTS = 100%

1) Find the following derivatives. Simplify completely unless you are told not to. Do not use logarithmic differentiation unless you are told to. (59 points total)

a) $D_\theta\left(\ln\left[\cos(5\theta)\right]\right)$ (5 points)

$$= \frac{1}{\cos(5\theta)} \cdot D_\theta\left[\cos(5\theta)\right] = \frac{1}{\cos(5\theta)} \cdot \left(\left[-\sin(5\theta)\right] \cdot [5]\right) = \boxed{-5\tan(5\theta)}$$

b) $D_x\left[\log_2\left(x^5+8\right)\right]$ (6 points)

$$= D_x\left[\frac{\ln\left(x^5+8\right)}{\ln(2)}\right] \text{ (by Change-of-Base Property of Logs) } = \frac{1}{\ln(2)} \cdot D_x\left[\ln\left(x^5+8\right)\right]$$

$$= \frac{1}{\ln(2)} \cdot \frac{1}{x^5+8} \cdot D_x\left(x^5+8\right) = \frac{1}{\ln(2)} \cdot \frac{1}{x^5+8} \cdot 5x^4 = \boxed{\frac{5x^4}{\left(x^5+8\right)\ln(2)}}$$

c) $D_x\left[\dfrac{\ln\left(6x^2-x\right)}{5e^{9x}+3}\right]$ (8 points)

You do not have to algebraically simplify, though perform all arithmetic.

(Use the Quotient Rule for Derivatives.)

$$= \frac{\text{Lo} \cdot \text{D}(\text{Hi}) - \text{Hi} \cdot \text{D}(\text{Lo})}{(\text{Lo})^2}$$

$$= \frac{\left[5e^{9x}+3\right] \cdot D_x\left[\ln\left(6x^2-x\right)\right] - \left[\ln\left(6x^2-x\right)\right] \cdot D_x\left[5e^{9x}+3\right]}{\left(5e^{9x}+3\right)^2}$$

$$= \frac{\left[5e^{9x}+3\right] \cdot \left[\frac{1}{6x^2-x}\right] \cdot \left[D_x\left(6x^2-x\right)\right] - \left[\ln\left(6x^2-x\right)\right] \cdot \left[5 \cdot 9e^{9x}\right]}{\left(5e^{9x}+3\right)^2}$$

$$= \boxed{\frac{\left[5e^{9x}+3\right] \cdot \left[\frac{1}{6x^2-x}\right] \cdot \left[12x-1\right] - \left[\ln\left(6x^2-x\right)\right] \cdot \left[45e^{9x}\right]}{\left(5e^{9x}+3\right)^2}, \text{ or } \frac{\left[5e^{9x}+3\right]\left[12x-1\right] - 45e^{9x}\left(6x^2-x\right)\left[\ln\left(6x^2-x\right)\right]}{\left(5e^{9x}+3\right)^2\left(6x^2-x\right)}}$$

if we multiply the numerator and the denominator by $\left(6x^2-x\right)$

d) $D_x\left(\left[\ln(2x+1)\right]^9\right)$ (5 points)

WARNING: The 9 **cannot** be "smacked down" by the Power Rule for Logs because the sole "ln" is included in the base that the 9 operates on.

$$= 9\left[\ln(2x+1)\right]^8 \cdot D_x\left[\ln(2x+1)\right] \quad \text{(by the Generalized Power Rule of Differentiation)}$$

$$= 9\left[\ln(2x+1)\right]^8 \cdot \frac{1}{2x+1} \cdot D_x(2x+1) = 9\left[\ln(2x+1)\right]^8 \cdot \frac{1}{2x+1} \cdot 2 = \boxed{\frac{18\left[\ln(2x+1)\right]^8}{2x+1}}$$

e) $D_x\left[\dfrac{x^5 \sec(x)}{(4x+\pi)^9}\right]$ (17 points)

You <u>must</u> use logarithmic differentiation and apply appropriate laws of logarithms whenever they apply, as in class. You do <u>not</u> have to write your final answer as a single fraction.

Let $y = \dfrac{x^5 \sec(x)}{(4x+\pi)^9} \Rightarrow$

$$\ln(y) = \ln\left[\frac{x^5 \sec(x)}{(4x+\pi)^9}\right] \quad \text{(We will expand the right side using Log Laws.)}$$

$$\ln(y) = \ln\left(x^5\right) + \ln\left[\sec(x)\right] - \ln\left[(4x+\pi)^9\right] \quad \text{(by Product, Quotient Rules of Logs)}$$

$$\ln(y) = 5\ln(x) + \ln\left[\sec(x)\right] - 9\ln(4x+\pi) \quad \text{(by Power Rule of Logs)} \Rightarrow$$

Use Implicit Differentiation to D_x both sides.

$$D_x\left[\ln(y)\right] = D_x\left[5\ln(x) + \ln\left[\sec(x)\right] - 9\ln(4x+\pi)\right]$$

$$D_x\left[\ln(y)\right] = 5 \cdot D_x\left[\ln(x)\right] + D_x\left(\ln\left[\sec(x)\right]\right) - 9 \cdot D_x\left[\ln(4x+\pi)\right]$$

$$\frac{1}{y} \cdot y' = 5 \cdot \frac{1}{x} + \frac{1}{\sec(x)} \cdot D_x\left[\sec(x)\right] - 9 \cdot \frac{1}{4x+\pi} \cdot D_x(4x+\pi)$$

$$\frac{y'}{y} = \frac{5}{x} + \frac{1}{\underbrace{\cancel{\sec(x)}}_{(1)}} \cdot \left[\overbrace{\cancel{\sec(x)}}^{(1)} \tan(x)\right] - \frac{9}{4x+\pi} \cdot 4$$

$$\frac{y'}{y} = \frac{5}{x} + \tan(x) - \frac{36}{4x+\pi}$$

Multiply both sides by y, and expand y in terms of x.

$$y' = \left[\frac{5}{x} + \tan(x) - \frac{36}{4x+\pi}\right] y$$

$$y' = \boxed{\left[\frac{5}{x} + \tan(x) - \frac{36}{4x+\pi}\right]\left[\frac{x^5 \sec(x)}{(4x+\pi)^9}\right], \text{ or } \frac{\left[x^4 \sec(x)\right]\left[x(4x+\pi)\tan(x) - 16x + 5\pi\right]}{(4x+\pi)^{10}}}$$

f) $D_x\left[2^{\ln(x)}\right]$ (6 points)

Answer only is fine, though logarithmic differentiation may help.

$$= \left[2^{\ln(x)}\ln(2)\right]\cdot\left(D_x\left[\ln(x)\right]\right) = \left[2^{\ln(x)}\ln(2)\right]\cdot\frac{1}{x} = \boxed{\frac{2^{\ln(x)}\ln(2)}{x}}$$

g) $D_x\left(x^{2x}\right)$ (12 points)

You <u>must</u> use logarithmic differentiation.

You do <u>not</u> have to write your final answer as a single fraction.

We want to differentiate something of the form $[f(x)]^{g(x)}$, where f and g are nonconstant, so let's use Logarithmic Differentiation to help bring that exponent down to earth!

$$\text{Let } y = x^{2x} \Rightarrow$$

$$\ln(y) = \ln\left(x^{2x}\right)$$

$$\ln(y) = 2x\ln(x) \quad (\text{by Power Rule of Logs}) \Rightarrow$$

Use Implicit Differentiation to D_x both sides.

$$D_x\left[\ln(y)\right] = D_x\left[2x\ln(x)\right]$$

$$\frac{1}{y}\cdot y' = \left[D_x(2x)\right]\cdot\left[\ln(x)\right] + \left[2x\right]\cdot\left(D_x\left[\ln(x)\right]\right)$$

(by the Product Rule for Derivatives)

$$\frac{y'}{y} = [2]\left[\ln(x)\right] + [2x]\left[\frac{1}{x}\right]$$

$$\frac{y'}{y} = 2\ln(x)+2$$

Multiply both sides by y, and expand y in terms of x.

$$y' = \left[2\ln(x)+2\right]y$$

$$y' = \boxed{\left[2\ln(x)+2\right]x^{2x}, \text{ or } 2x^{2x}\left[\ln(x)+1\right]}$$

2) Evaluate the following integrals. Simplify completely. (46 points total)

a) $\int_1^2 \left(x^2\right)\left(3^{x^3+1}\right)dx$ (11 points)

Give an exact answer; do not approximate.

<u>Note</u>: The integrand is continuous on $[1, 2]$, so we may apply the Fundamental Theorem of Calculus (FTC), Part II directly.

Method 1 (Change the limits of integration.)

Let $u = x^3 + 1 \quad \Rightarrow$

$$du = 3x^2\,dx \quad \Rightarrow \quad \left(\text{Can use: } x^2\,dx = \frac{1}{3}du\right)$$

Change the limits of integration:

$$x = 1 \quad \Rightarrow \quad u = (1)^3 + 1 = 2 \quad \Rightarrow \quad u = 2$$

$$x = 2 \quad \Rightarrow \quad u = (2)^3 + 1 = 9 \quad \Rightarrow \quad u = 9$$

$$\int_1^2 \left(x^2\right)\left(3^{x^3+1}\right) dx$$

$$= \frac{1}{3}\int_1^2 \left(3x^2\right)\left(3^{x^3+1}\right) dx \quad \text{(by Compensation), or}$$

$$\int_1^2 \left(3^{x^3+1}\right)\left(x^2\,dx\right), \text{ or } \int_{u=2}^{u=9} \left(3^u\right)\left(\frac{1}{3}du\right)$$

$$= \frac{1}{3}\int_2^9 3^u\,du = \frac{1}{3}\left[\frac{3^u}{\ln(3)}\right]_2^9 = \frac{1}{3\ln(3)}\left[3^u\right]_2^9$$

$$= \frac{1}{3\ln(3)}\left[3^{(9)} - 3^{(2)}\right] = \frac{1}{3\ln(3)}(19{,}683 - 9) = \frac{19{,}674}{3\ln(3)} = \boxed{\frac{6558}{\ln(3)}}$$

Method 2 (Work out the corresponding indefinite integral first.)

Let $u = x^3 + 1 \quad \Rightarrow$

$$du = 3x^2\,dx \quad \Rightarrow \quad \left(\text{Can use: } x^2\,dx = \frac{1}{3}du\right)$$

$$\int \left(x^2\right)\left(3^{x^3+1}\right) dx = \frac{1}{3}\int \left(3x^2\right)\left(3^{x^3+1}\right) dx \quad \text{(by Compensation), or}$$

$$\int \left(3^{x^3+1}\right)\left(x^2\,dx\right), \text{ or } \int \left(3^u\right)\left(\frac{1}{3}du\right)$$

$$= \frac{1}{3}\int 3^u\,du = \frac{1}{3}\left[\frac{3^u}{\ln(3)}\right] + C = \frac{3^{x^3+1}}{3\ln(3)} + C$$

Now, apply the FTC directly using our antiderivative (where $C = 0$).

$$\int_1^2 \left(x^2\right)\left(3^{x^3+1}\right) dx = \left[\frac{3^{x^3+1}}{3\ln(3)}\right]_1^2 = \frac{1}{3\ln(3)}\left[3^{x^3+1}\right]_1^2 = \frac{1}{3\ln(3)}\left(\left[3^{(2)^3+1}\right] - \left[3^{(1)^3+1}\right]\right)$$

$$= \frac{1}{3\ln(3)}\left(3^9 - 3^2\right) = \frac{1}{3\ln(3)}(19{,}683 - 9) = \frac{19{,}674}{3\ln(3)} = \boxed{\frac{6558}{\ln(3)}}$$

b) $\displaystyle\int \frac{9x}{x^2-4}\,dx$ (8 points)

Let $u = x^2 - 4 \Rightarrow$

$du = 2x\,dx \Rightarrow \left(\text{Can use: } x\,dx = \frac{1}{2}du\right)$

$$\int \frac{9x}{x^2-4}\,dx = \int \frac{9\cdot\frac{1}{2}du}{u}, \text{ or } 9\cdot\frac{1}{2}\int\frac{2x}{x^2-4}\,dx \text{ (by Compensation)}$$

$$= \frac{9}{2}\int\frac{du}{u} = \frac{9}{2}\ln|u| + C = \boxed{\frac{9}{2}\ln\left|x^2-4\right| + C, \text{ or } \ln\left(\left|x^2-4\right|^{9/2}\right)+C}$$

Note 1: $x^2 - 4$ is negative for some real values of x, so keep the absolute value symbols.

Note 2: You can check that the derivative of our answer is the integrand.

c) $\displaystyle\int \frac{1}{x\left[\ln(x)\right]^4}\,dx$ (7 points)

Let $u = \ln(x) \Rightarrow$

$du = \frac{1}{x}dx$

$$\int \frac{1}{x\left[\ln(x)\right]^4}\,dx = \int \frac{1}{\left[\ln(x)\right]^4}\cdot\left[\frac{1}{x}\,dx\right]$$

$$= \int \frac{1}{u^4}\,du$$

$$= \int u^{-4}\,du$$

$$= \frac{u^{-3}}{-3} + C$$

$$= -\frac{1}{3u^3} + C$$

$$= \boxed{-\frac{1}{3\left[\ln(x)\right]^3} + C, \text{ or } C - \frac{1}{3\left[\ln(x)\right]^3}}$$

d) $\int \frac{\sec\left(\sqrt{\theta}\right)}{\sqrt{\theta}}\,d\theta$ (10 points)

Let $u = \sqrt{\theta}$

$$u = \theta^{1/2} \quad \Rightarrow$$

$$du = \frac{1}{2}\theta^{-1/2}\,d\theta$$

$$du = \frac{1}{2\sqrt{\theta}}\,d\theta \quad \Rightarrow \quad \left(\text{Can use: } \frac{1}{\sqrt{\theta}}\,d\theta = 2\,du\right)$$

$$\int \frac{\sec\left(\sqrt{\theta}\right)}{\sqrt{\theta}}\,d\theta = \int\left[\sec\left(\sqrt{\theta}\right)\right]\cdot\left[\frac{1}{\sqrt{\theta}}\,d\theta\right] \quad \text{or} \quad 2\int \frac{\sec\left(\sqrt{\theta}\right)}{2\sqrt{\theta}}\,d\theta \quad \text{(by Compensation)}$$

$$= \int \sec(u)\cdot 2\,du \qquad \text{or} \quad 2\int\left[\sec\left(\sqrt{\theta}\right)\right]\cdot\left[\frac{1}{2\sqrt{\theta}}\,d\theta\right]$$

$$= 2\int \sec(u)\,du$$

$$= 2\ln\left|\sec(u) + \tan(u)\right| + C$$

$$= \boxed{2\ln\left|\sec\left(\sqrt{\theta}\right) + \tan\left(\sqrt{\theta}\right)\right| + C, \text{ or } \ln\left(\left[\sec\left(\sqrt{\theta}\right) + \tan\left(\sqrt{\theta}\right)\right]^2\right) + C}$$

e) $\int \csc(x)\,dx$ (4 points)

Answer only is fine.

$$\int \csc(x)\,dx = \boxed{\ln\left|\csc(x) - \cot(x)\right| + C, \text{ or } -\ln\left|\csc(x) + \cot(x)\right| + C}$$

f) $\int \cot(7x)\,dx$ (6 points)

Answer only is fine.

$$\int \cot(7x)\,dx = \boxed{\frac{1}{7}\ln\left|\sin(7x)\right| + C}$$

<u>Showing the *u*-sub (Optional):</u>

$$u = 7x \quad \Rightarrow$$

$$du = 7\,dx \quad \Rightarrow \quad \left(\text{Can use: } dx = \frac{1}{7}\,du\right)$$

$$\int \cot(7x)\,dx = \frac{1}{7}\int\left[7\cot(7x)\right]dx \ \text{(Compensation)} = \frac{1}{7}\int\left[\cot(7x)\right]\cdot\left[7\,dx\right]$$

$$= \frac{1}{7}\int \cot(u)\,du = \frac{1}{7}\ln\left|\sin(u)\right| + C = \boxed{\frac{1}{7}\ln\left|\sin(7x)\right| + C}$$