SOLUTIONS TO THE FINAL - PART 1

MATH 150 – FALL 2016 – KUNIYUKI
PART 1: 135 POINTS, PART 2: 115 POINTS, TOTAL: 250 POINTS

No notes, books, or calculators allowed.

135 points: 45 problems, 3 pts. each. You do not have to algebraically simplify or box in your answers, unless you are instructed to. Fill in all blanks after "=" signs.

DERIVATIVES (66 POINTS TOTAL)

$$D_x\left(x^{\sqrt{2}}\right) = \sqrt{2}x^{\sqrt{2}-1}$$

$$\begin{aligned} D_x\left[x^3\cos(x)\right] &= \left[D_x\left(x^3\right)\right]\cdot\left[\cos(x)\right] + \left[x^3\right]\cdot\left(D_x\left[\cos(x)\right]\right) \text{ (by Product Rule)} \\ &= \left[3x^2\right]\left[\cos(x)\right] + \left[x^3\right]\left[-\sin(x)\right] \\ &= 3x^2\cos(x) - x^3\sin(x), \text{ or } x^2\left[3\cos(x) - x\sin(x)\right] \end{aligned}$$

$$\begin{aligned} D_x\left(\frac{x^7}{2x-5}\right) &= \frac{\left[2x-5\right]\cdot\left[D_x\left(x^7\right)\right] - \left[x^7\right]\cdot\left[D_x(2x-5)\right]}{(2x-5)^2} \text{ (by Quotient Rule)} \\ &= \frac{\left[2x-5\right]\cdot\left[7x^6\right] - \left[x^7\right]\cdot\left[2\right]}{(2x-5)^2}, \text{ or } \frac{12x^7-35x^6}{(2x-5)^2}, \text{ or } \frac{x^6(12x-35)}{(2x-5)^2} \end{aligned}$$

$$\begin{aligned} D_x\left[\left(e^x+4\right)^6\right] &= \left[6\left(e^x+4\right)^5\right]\cdot\left[D_x\left(e^x+4\right)\right] = \left[6\left(e^x+4\right)^5\right]\cdot\left[e^x\right] \\ &= 6e^x\left(e^x+4\right)^5 \text{ (by Gen. Power Rule)} \end{aligned}$$

$$D_x\left[\tan(x)\right] = \sec^2(x)$$

$$D_x\left[\cot(x)\right] = -\csc^2(x)$$

$$D_x\left[\sec(x)\right] = \sec(x)\tan(x)$$

$$D_x\left[\csc(x)\right] = -\csc(x)\cot(x)$$

$$\begin{aligned} D_x\left[\sin(4x+7)\right] &= \left[\cos(4x+7)\right]\cdot\left[D_x(4x+7)\right] = \left[\cos(4x+7)\right]\cdot\left[4\right] \\ &= 4\cos(4x+7) \text{ (by Gen. Trig Rule)} \end{aligned}$$

$$D_x\left(e^{\frac{1}{3}x}\right) = \left[e^{\frac{1}{3}x}\right]\cdot\left[D_x\left(\frac{1}{3}x\right)\right] = \left[e^{\frac{1}{3}x}\right]\cdot\left[\frac{1}{3}\right] = \frac{1}{3}e^{\frac{1}{3}x}$$

MORE!

$$D_x\left(5^x\right) = 5^x\ln(5)$$

$$D_x\left(10^{x^3}\right) = \left[10^{x^3}\ln(10)\right]\cdot\left[D_x\left(x^3\right)\right] = \left[10^{x^3}\ln(10)\right]\cdot\left[3x^2\right] = 3x^2\cdot 10^{x^3}\ln(10)$$

$$D_x\left[\ln\left(7x^2+1\right)\right] = \left[\frac{1}{7x^2+1}\right]\cdot\left[D_x\left(7x^2+1\right)\right] = \left[\frac{1}{7x^2+1}\right]\cdot\left[14x\right] = \frac{14x}{7x^2+1}$$

$$D_x\left[\log_9(x)\right] = D_x\left[\frac{\ln(x)}{\ln(9)}\right] = \left[\frac{1}{\ln(9)}\right]\cdot\left(D_x\left[\ln(x)\right]\right) = \left[\frac{1}{\ln(9)}\right]\cdot\left[\frac{1}{x}\right] = \frac{1}{x\ln(9)}$$

$$D_x\left[\sin^{-1}(x)\right] = \frac{1}{\sqrt{1-x^2}}$$

$$D_x\left[\cos^{-1}(x)\right] = -\frac{1}{\sqrt{1-x^2}}$$

$$D_x\left[\tan^{-1}(x)\right] = \frac{1}{1+x^2}$$

$$D_x\left[\sec^{-1}(x)\right] = \frac{1}{x\sqrt{x^2-1}}$$ (Assume the usual range for $\sec^{-1}(x)$ in our class.)

$$D_x\left[\tan^{-1}(7x)\right] = \left[\frac{1}{1+(7x)^2}\right]\cdot\left[D_x(7x)\right] = \left[\frac{1}{1+(7x)^2}\right]\cdot[7] = \frac{7}{1+49x^2}$$

$$D_x\left[\sinh(x)\right] = \cosh(x)$$

$$D_x\left[\cosh(x)\right] = \sinh(x)$$

$$D_x\left[\operatorname{sech}(x)\right] = -\operatorname{sech}(x)\tanh(x)$$

INDEFINITE INTEGRALS (42 POINTS TOTAL)

$$\int x^9\,dx = \frac{x^{10}}{10} + C$$

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int e^{-7x}\,dx = \frac{e^{-7x}}{-7} + C = -\frac{e^{-7x}}{7} + C, \text{ or } C - \frac{1}{7e^{7x}}$$

$$\int 6^x\,dx = \frac{6^x}{\ln(6)} + C$$

$$\int \sin(x)\,dx = -\cos(x) + C$$

$$\int \tan(x)\,dx = -\ln|\cos(x)| + C, \text{ or } \ln|\sec(x)| + C$$

$$\int \cot(x)\,dx = \ln|\sin(x)| + C$$

$$\int \sec(x)\,dx = \ln|\sec(x) + \tan(x)| + C$$

$$\int \csc(x)\,dx = \ln|\csc(x) - \cot(x)| + C, \text{ or } -\ln|\csc(x) + \cot(x)| + C$$

$$\int \cos(3x)\,dx = \frac{1}{3}\sin(3x) + C$$

$$\int \sec^2(x)\,dx = \tan(x) + C$$

$$\int \frac{1}{36+x^2}\,dx = \frac{1}{6}\tan^{-1}\left(\frac{x}{6}\right) + C$$

$$\int \frac{1}{\sqrt{36-x^2}}\,dx = \sin^{-1}\left(\frac{x}{6}\right) + C$$

$$\int \cosh(x)\,dx = \sinh(x) + C$$

WARNING: YOU'VE BEEN DEALING WITH INDEFINITE INTEGRALS.
DID YOU FORGET SOMETHING? (+ C)

INVERSE TRIGONOMETRIC FUNCTIONS (6 POINTS TOTAL)

- $\lim_{x\to\infty} \tan^{-1}(x) = \boxed{\dfrac{\pi}{2}}$ (Drawing a graph may help.)

- If $f(x) = \sin^{-1}(x)$, what is the range of f in interval form (the form with parentheses and/or brackets)? $\text{Range}(f) = \boxed{\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]}$.

HYPERBOLIC FUNCTIONS (6 POINTS TOTAL)

- The definition of $\sinh(x)$ (as given in class) is: $\sinh(x) = \boxed{\dfrac{e^x - e^{-x}}{2}}$

- Complete the following identity: $\cosh^2(x) - \sinh^2(x) = \boxed{1}$
  (We mentioned this identity in class.)

TRIGONOMETRIC IDENTITIES (15 POINTS TOTAL)

Complete each of the following identities, based on the type of identity given.

- $\tan^2(x) + 1 = \sec^2(x)$ (Pythagorean Identity)

- $\cos(-x) = \cos(x)$ (Even/Odd Identity)

- $\sin(2x) = 2\sin(x)\cos(x)$ (Double-Angle Identity)

- $\cos(2x) = \cos^2(x) - \sin^2(x)$, or $1 - 2\sin^2(x)$, or $2\cos^2(x) - 1$
  (Double-Angle Identity)
  (For $\cos(2x)$, I gave you three versions; you may pick any one.)

- $\sin^2(x) = \dfrac{1-\cos(2x)}{2}$ (Power-Reducing Identity)

SOLUTIONS TO THE FINAL - PART 2

MATH 150 – FALL 2016 – KUNIYUKI
PART 1: 135 POINTS, PART 2: 115 POINTS, TOTAL: 250 POINTS
A scientific calculator and an appropriate sheet of notes are allowed on this final part.

1) Find the following limits. Each answer will be a real number, ∞, $-\infty$, or DNE (Does Not Exist). Write ∞ or $-\infty$ when appropriate. If a limit does not exist, and ∞ and $-\infty$ are inappropriate, write "DNE." **Box in your final answers.** (14 points total)

a) $\lim\limits_{x\to\infty}\dfrac{\cos\left(x^3\right)}{x^4}$ **Show all work, as in class.** (6 points)

$\lim\limits_{x\to\infty}\dfrac{\cos\left(x^3\right)}{x^4}=\boxed{0}$. Prove this using the Sandwich / Squeeze Theorem:

$$-1\le\cos\left(x^3\right)\le 1 \quad \left(\forall x\in\mathbb{R}\right)$$

Observe that $x^4>0,\ \forall x>0$ $\left(\text{we may assume this, since we let } x\to\infty\right)$.
Divide all three parts by x^4.

$$\text{As } x\to\infty,\ \underbrace{-\frac{1}{x^4}}_{\to 0}\le\underbrace{\frac{\cos\left(x^3\right)}{x^4}}_{\substack{\text{So, }\to 0\\ \text{by the Sandwich/}\\ \text{Squeeze Theorem}}}\le\underbrace{\frac{1}{x^4}}_{\to 0}\quad\left(\forall x>0\right)$$

More precisely: $\lim\limits_{x\to\infty}\left(-\dfrac{1}{x^4}\right)=0$, and $\lim\limits_{x\to\infty}\dfrac{1}{x^4}=0$.

Therefore, by the Sandwich / Squeeze Theorem, $\lim\limits_{x\to\infty}\dfrac{\cos\left(x^3\right)}{x^4}=0$.

b) $\lim\limits_{x\to -2^+}\dfrac{x}{x^2-3x-10}$ **Show all work, as in class.** (6 points)

$$\lim_{x\to -2^+}\frac{x}{x^2-3x-10}=\lim_{x\to -2^+}\frac{x}{\underbrace{(x+2)}_{\to 0^+}\underbrace{(x-5)}_{\to -7}}\ \left(\text{Limit Form }\frac{-2}{0^-}\right)=\boxed{\infty}$$

c) $\lim\limits_{r\to\infty}\dfrac{r^3+1}{\left(r^5+r\right)^2}$ **Answer only is fine.** (2 points)

$\lim\limits_{r\to\infty}\dfrac{r^3+1}{\left(r^5+r\right)^2}=\boxed{0}$, because we are taking a "long-run" limit of a proper ("bottom-heavy") rational expression as $r\to\infty$. The degree of the denominator (10, since the leading term would be r^{10} after expanding the square) is greater than the degree of the numerator (3).

2) Let $f(x)=\begin{cases} x+4, & x\neq 3 \\ 9, & x=3 \end{cases}$. Classify the discontinuity at $x=3$. Box in one: (2 points)

Infinite discontinuity   Jump discontinuity $\boxed{\text{Removable discontinuity}}$

$\lim\limits_{x\to 3} f(x) = \lim\limits_{x\to 3}(x+4) = 3+4 = 7 \neq 9$, which is $f(3)$. (The limit value and the function value exist but are unequal.) If $f(3)$ were 7, then f would be continuous at 3.

3) Use the limit definition of the derivative to prove that $D_x\left(5x^2-x+7\right)=10x-1$, $\forall x\in\mathbb{R}$. Do **not** use derivative short cuts we have used in class. (10 points)

Let $f(x)=5x^2-x+7$.

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\left[5(x+h)^2-(x+h)+7\right]-\left[5x^2-x+7\right]}{h}$$

$$= \lim_{h\to 0}\frac{\left[5\left(x^2+2xh+h^2\right)-x-h+7\right]-\left[5x^2-x+7\right]}{h}$$

$$= \lim_{h\to 0}\frac{\cancel{5x^2}+10xh+5h^2\cancel{-x}-h\cancel{+7}\cancel{-5x^2}\cancel{+x}\cancel{-7}}{h}$$

$$= \lim_{h\to 0}\frac{10xh+5h^2-h}{h} = \lim_{h\to 0}\frac{\overset{(1)}{\cancel{h}}(10x+5h-1)}{\underset{(1)}{\cancel{h}}} = \lim_{h\to 0}(10x+5h-1)$$

$$= 10x+5(0)-1 = 10x-1.$$ (Q.E.D.)

4) Let $f(x)=\log_4(x)$. Consider the graph of $y=f(x)$ in the usual *xy*-plane. Find a Point-Slope Form of the **equation** of the tangent line to the graph at the point where $x=16$. Give exact values; you do not have to approximate. (8 points)

$f(16) = \log_4(16) = 2$, so the point of interest is $(16, 2)$.

Find $f'(16)$, the slope m of the tangent line to the graph at that point.

$$f'(x) = D_x\left[\log_4(x)\right] = D_x\left[\frac{\ln(x)}{\ln(4)}\right]$$ (Change-of-Base Property of Logarithms)

$$= \left[\frac{1}{\ln(4)}\right]\cdot\left(D_x\left[\ln(x)\right]\right) = \left[\frac{1}{\ln(4)}\right]\cdot\left[\frac{1}{x}\right] = \frac{1}{x\ln(4)} \Rightarrow$$

$$f'(16) = \frac{1}{16\ln(4)}, \text{ or } \frac{1}{\ln\left(4^{16}\right)}$$ (This is our desired slope, *m*.)

Find a Point-Slope Form of the equation of the tangent line.

$$y-y_1=m(x-x_1) \Rightarrow \boxed{y-2=\frac{1}{16\ln(4)}(x-16)}$$

5) Assume that x and y are differentiable functions of t. Evaluate $D_t\left(e^{xy}\right)$ when $x=3$, $y=5$, $\frac{dx}{dt}=2$, and $\frac{dy}{dt}=-4$. (8 points)

$$D_t\left(e^{xy}\right) = \left[e^{xy}\right]\cdot\left[D_t\left(xy\right)\right] = \left[e^{xy}\right]\cdot\left[\frac{dx}{dt}\cdot y + x\cdot\frac{dy}{dt}\right] \text{ (by the Product Rule)}$$

$$= \left[e^{(3)(5)}\right]\cdot\left[(2)(5) + (3)(-4)\right] = \left[e^{15}\right]\cdot\left[10-12\right] = \left[e^{15}\right]\cdot\left[-2\right] = \boxed{-2e^{15}}$$

6) Let $f(x)=x^3-2x^2-4x+1$. (14 points total)

a) Find the two critical numbers of f.

$f'(x) = 3x^2-4x-4 = (3x+2)(x-2)$, which is never undefined ("DNE") but is 0 at $x=-\frac{2}{3}$ and $x=2$. These numbers are in $\text{Dom}(f)$, which is $\mathbb{R}$, so the critical numbers are: $\boxed{-\frac{2}{3} \text{ and } 2}$.

<u>Note</u>: We could use the Quadratic Formula (QF) with $a=3$, $b=-4$, and $c=-4$.

$$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{-(-4)\pm\sqrt{(-4)^2-4(3)(-4)}}{2(3)} = \frac{4\pm\sqrt{16+48}}{2(3)} = \frac{4\pm\sqrt{64}}{6} = \frac{4\pm 8}{6} = \frac{2\pm 4}{3} \Rightarrow$$

$$x = \frac{2+4}{3} = \frac{6}{3} = 2 \quad \text{or} \quad x = \frac{2-4}{3} = \frac{-2}{3} = -\frac{2}{3}$$

b) Consider the graph of $y=f(x)$, although you do not have to draw it. Use the First Derivative Test to classify the point at $x=2$ as a local maximum point, a local minimum point, or neither.

f is continuous on $\mathbb{R}$, so the First Derivative Test (1st DT) should apply wherever we have critical numbers (CNs). Both f and f' are continuous on $\mathbb{R}$, so use just the CNs as "fenceposts" on the real number line where f' could change sign.

	(not needed)	**−2 / 3**	Test $x=0$	**2**	Test $x=3$
f' sign (see below)		0	−	0	+
f			$\searrow$		$\nearrow$
Classify point at CN (1st DT)				**L.Min. Pt.**	

$f'(x)=(3x+2)(x-2)$

$f'(0)= \quad (+) \quad (-) \quad = \quad -$ **or** Evaluate f' at 0 and 3 directly.

$f'(3)= \quad (+) \quad (+) \quad = \quad +$

Also, the graph of $y=f'(x)$ is an upward-opening parabola with two distinct x-intercepts at $\left(-\frac{2}{3},0\right)$ and $(2,0)$. The multiplicities of the zeros of f' are both odd (1), so signs alternate in our "windows." Answer: $\boxed{\text{Local Minimum Point}}$

c) Use the Second Derivative Test to classify the point at the **other** critical number as a local maximum point or a local minimum point.

$f'\left(-\frac{2}{3}\right) = 0$, so we may apply the Second Derivative Test for $x = -\frac{2}{3}$.

$$f'(x) = 3x^2 - 4x - 4 \quad \Rightarrow$$
$$f''(x) = 6x - 4 \quad \Rightarrow$$
$$f''\left(-\frac{2}{3}\right) = 6\left(-\frac{2}{3}\right) - 4 = -8 \quad \Rightarrow \quad f''\left(-\frac{2}{3}\right) < 0$$

Therefore, the point at $x = -\frac{2}{3}$ is a $\boxed{\text{Local Maximum Point}}$.

Think: Concave down ($\cap$) "at" (actually, on a neighborhood of) $x = -\frac{2}{3}$.

7) Evaluate the following integrals. (20 points total)

a) $\int_9^{16} \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$. Give an exact answer. (10 points)

Let $u = \sqrt{x}$ or $x^{1/2}$ $\Rightarrow$

$$du = \frac{1}{2}x^{-1/2}\,dx \quad \Rightarrow \quad du = \frac{1}{2\sqrt{x}}\,dx \quad \Rightarrow \quad \left(\text{Can use: } \frac{1}{\sqrt{x}}\,dx = 2\,du\right)$$

Method 1 (Change the limits of integration.)

$$x = 9 \quad \Rightarrow \quad u = \sqrt{9} = 3 \quad \Rightarrow \quad u = 3$$
$$x = 16 \quad \Rightarrow \quad u = \sqrt{16} = 4 \quad \Rightarrow \quad u = 4$$

$$\int_9^{16} \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = 2\int_9^{16} \frac{e^{\sqrt{x}}}{2\sqrt{x}}\,dx \;(\text{by Compensation}) = 2\int_9^{16} e^{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}}\,dx$$
$$= 2\int_3^4 e^u\,du = 2\left[e^u\right]_3^4 = \boxed{2\left(e^4 - e^3\right), \text{ or } 2e^3(e-1)}$$

Method 2 (Work out the corresponding indefinite integral first.)

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = 2\int \frac{e^{\sqrt{x}}}{2\sqrt{x}}\,dx \;(\text{by Compensation}) = 2\int e^{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}}\,dx$$
$$= 2\int e^u\,du = 2e^u + C = 2e^{\sqrt{x}} + C$$

Now, apply the FTC directly using our antiderivative (where $C = 0$).

$$\int_9^{16} \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = \left[2e^{\sqrt{x}}\right]_9^{16} = 2\left[e^{\sqrt{x}}\right]_9^{16} = 2\left(\left[e^{\sqrt{16}}\right] - \left[e^{\sqrt{9}}\right]\right)$$
$$= \boxed{2\left(e^4 - e^3\right), \text{ or } 2e^3(e-1)}$$

b) $\displaystyle\int \frac{7x}{x^4+81}\,dx$ (10 points)

Hint: Consider the Chapter 8 material on inverse trigonometric functions!

Use the template: $\displaystyle\int \frac{1}{u^2+81}\,du$, or $\displaystyle\int \frac{du}{81+u^2} = \frac{1}{9}\tan^{-1}\left(\frac{u}{9}\right)+C$.

$$\text{Let} \quad u = x^2 \quad \Rightarrow \quad u^2 = x^4$$

$$du = 2x\,dx \quad \Rightarrow \quad \left(\text{Can use: } x\,dx = \frac{1}{2}\,du\right)$$

$$\int \frac{7x}{x^4+81}\,dx = 7\int \frac{x}{81+x^4}\,dx = 7\cdot\frac{1}{2}\int \frac{2x}{81+x^4}\,dx \ \left(\text{by Compensation}\right)$$

$$= \frac{7}{2}\int \frac{du}{81+u^2} = \frac{7}{2}\left[\frac{1}{9}\tan^{-1}\left(\frac{u}{9}\right)\right]+C = \boxed{\frac{7}{18}\tan^{-1}\left(\frac{x^2}{9}\right)+C}$$

Alternative Method (using basic template $\displaystyle\int \frac{1}{u^2+1}\,du$, or $\displaystyle\int \frac{du}{1+u^2} = \tan^{-1}(u)+C$):

$$\int \frac{7x}{x^4+81}\,dx = 7\int \frac{x}{81+x^4}\,dx = 7\int \frac{x}{81\left(1+\frac{x^4}{81}\right)}\,dx = \frac{7}{81}\int \frac{x}{1+\frac{x^4}{81}}\,dx$$

$$= \frac{7}{81}\int \frac{x}{1+\left(\frac{x^2}{9}\right)^2}\,dx$$

$$\left[\begin{array}{lll} \text{Let} \quad u = \frac{x^2}{9}, \text{ or } \frac{1}{9}x^2 & \Rightarrow & u^2 = \left(\frac{x^2}{9}\right)^2 = \frac{x^4}{81} \\ du = \frac{2}{9}x\,dx & \Rightarrow & \frac{9}{2}\,du = x\,dx \ \left(\text{or use Compensation}\right) \end{array}\right]$$

$$= \frac{7}{\cancel{81}_{(9)}}\cdot\frac{\cancel{9}^{(1)}}{2}\int \frac{\frac{2}{9}x}{1+\left(\frac{x^2}{9}\right)^2}\,dx = \frac{7}{18}\int \frac{du}{1+u^2} = \frac{7}{18}\tan^{-1}(u)+C = \boxed{\frac{7}{18}\tan^{-1}\left(\frac{x^2}{9}\right)+C}$$

8) The velocity function for a particle moving along a coordinate line $\left(\text{for } t>0\right)$ is given by $v(t)=\frac{1}{t^4}-\sqrt{t}$, where t is time measured in seconds and velocity is given in meters per second. The particle's position is measured in meters. Find $s(t)$, the corresponding position function [rule], if $s(1)=2$ (meters). (9 points)

$$\int v(t)\,dt = \int\left(\frac{1}{t^4}-\sqrt{t}\right)dt = \int\left(t^{-4}-t^{1/2}\right)dt \Rightarrow$$

$$s(t) = \frac{t^{-3}}{-3}-\frac{t^{3/2}}{3/2}+C = -\frac{1}{3t^3}-\frac{2}{3}t^{3/2}+C$$

Find C.

We know: $s(1)=2$ (meters).

$$s(1)=-\frac{1}{3(1)^3}-\frac{2}{3}(1)^{3/2}+C$$

$$2=-\frac{1}{3}-\frac{2}{3}+C$$

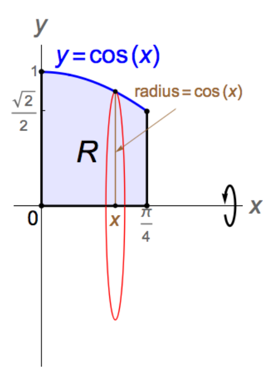
$$2=-1+C$$

$$C=3 \Rightarrow$$

$$\boxed{s(t)=-\frac{1}{3t^3}-\frac{2}{3}t^{3/2}+3,\ \text{or}\ 3-\frac{1}{3t^3}-\frac{2}{3}t\sqrt{t},\ \text{or}\ \frac{9t^3-1-2t^4\left(\sqrt{t}\right)}{3t^3},\ \text{or}\ \frac{9t^3-1-2t^{9/2}}{3t^3}\ \text{(in meters)}}$$

9) The region R is bounded by the x-axis, the y-axis, and the graphs of $y=\cos(x)$ and $x=\frac{\pi}{4}$ in the usual xy-plane. **Sketch and shade in the region *R*.** Find the **volume** of the solid generated by revolving R about the x-axis. **Evaluate** your integral completely. Give an **exact** answer in simplest form with appropriate units. Distances and lengths are measured in meters. Hint: Use a Power-Reducing Identity. (18 points)

Let $f(x)=\cos(x)$. Then, f is nonnegative and continuous on the x-interval $\left[0,\frac{\pi}{4}\right]$.



The region R and the equation $y=\cos(x)$ suggest a "*dx* scan" and the Disk Method.

V, the volume of the solid, is given by:

$$V = \int_0^{\pi/4} \pi(\text{radius})^2\,dx = \int_0^{\pi/4} \pi\left[\cos(x)\right]^2 dx = \pi\int_0^{\pi/4}\cos^2(x)\,dx$$

$$= \pi\int_0^{\pi/4}\frac{1+\cos(2x)}{2} \quad \text{(by a Power-Reducing Identity)} \quad = \frac{\pi}{2}\int_0^{\pi/4}\left[1+\cos(2x)\right]dx$$

$$= \frac{\pi}{2}\left[x+\frac{1}{2}\sin(2x)\right]_0^{\pi/4} \quad \text{(by "Guess-and-check," or using } u=2x\text{)}$$

$$= \frac{\pi}{2}\left(\left[\frac{\pi}{4}+\frac{1}{2}\sin\left(2\cdot\frac{\pi}{4}\right)\right]-\left[0+\frac{1}{2}\sin(2\cdot 0)\right]\right)$$

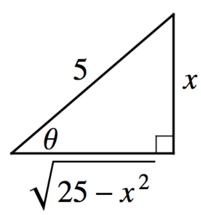
$$= \frac{\pi}{2}\left(\left[\frac{\pi}{4}+\frac{1}{2}\sin\left(\frac{\pi}{2}\right)\right]-\left[0+\frac{1}{2}\sin(0)\right]\right)$$

$$= \frac{\pi}{2}\left(\left[\frac{\pi}{4}+\frac{1}{2}(1)\right]-[0]\right) = \frac{\pi}{2}\left(\frac{\pi}{4}+\frac{1}{2}\right) = \frac{\pi}{2}\left(\frac{\pi+2}{4}\right) = \boxed{\frac{\pi(\pi+2)}{8}\ \text{m}^3}$$

10) Rewrite $\tan\left(\sin^{-1}\left(\frac{x}{5}\right)\right)$ as an algebraic expression in x, where $0<x<5$.

(7 points)

Let $\theta=\sin^{-1}\left(\frac{x}{5}\right)$ $(\theta \text{ acute})$ $\Rightarrow$ $\sin(\theta)=\frac{x}{5}$



← found by the Pythagorean Theorem

$$\tan(\theta)=\frac{\text{opp.}}{\text{adj.}}=\boxed{\frac{x}{\sqrt{25-x^2}}}$$

Note: Rationalizing the denominator is usually unnecessary if the radicand involved is variable.

11) Find $D_r\left[\sin^{-1}(\sinh(r))\right]$. (5 points)

Use the template: $D_r\left[\sin^{-1}(u)\right]=\frac{1}{\sqrt{1-u^2}}\cdot\left[D_r(u)\right]$.

$$D_r\left[\sin^{-1}(\sinh(r))\right] = \left[\frac{1}{\sqrt{1-\sinh^2(r)}}\right]\cdot\left[D_r(\sinh(r))\right] = \left[\frac{1}{\sqrt{1-\sinh^2(r)}}\right]\cdot\left[\cosh(r)\right]$$

$= \boxed{\frac{\cosh(r)}{\sqrt{1-\sinh^2(r)}}}$. Note: $1-\sinh^2(r)$ is **not** equivalent to $\cosh^2(r)$, but $1+\sinh^2(r)$ is.