SOLUTIONS TO THE FINAL - PART 1

MATH 150 – FALL 2016 – KUNIYUKI
PART 1: 135 POINTS, PART 2: 115 POINTS, TOTAL: 250 POINTS

No notes, books, or calculators allowed.

135 points: 45 problems, 3 pts. each. You do not have to algebraically simplify or box in your answers, unless you are instructed to. Fill in all blanks after “=” signs.

DERIVATIVES (66 POINTS TOTAL)

\[ D_x \left( x^{\sqrt{x}} \right) = \sqrt{2}x^{\frac{\sqrt{x}-1}{2}} \]

\[ D_x \left[ x^3 \cos(x) \right] = \left[ D_x \left( x^3 \right) \right] \cdot \cos(x) + \left[ x^3 \right] \cdot \left( D_x \left[ \cos(x) \right] \right) \] (by Product Rule)

\[ = \left[ 3x^2 \right] \cdot \cos(x) + \left[ x^3 \right] \cdot \left[ -\sin(x) \right] \]

\[ = 3x^2 \cos(x) - x^3 \sin(x), \text{ or } x^2 \left[ 3\cos(x) - x \sin(x) \right] \]

\[ D_x \left( \frac{x^7}{2x-5} \right) = \frac{[2x-5] \cdot \left[ D_x \left( x^7 \right) \right] - \left[ x^7 \right] \cdot \left[ D_x \left( 2x-5 \right) \right]}{(2x-5)^2} \] (by Quotient Rule)

\[ = \frac{[2x-5] \cdot [7x^6] - \left[ x^7 \right] \cdot [2]}{(2x-5)^2}, \text{ or } \frac{12x^7 - 35x^6}{(2x-5)^2}, \text{ or } \frac{x^6 \left( 12x - 35 \right)}{(2x-5)^2} \]

\[ D_x \left[ (e^x + 4)^6 \right] = \left[ 6(e^x + 4)^5 \right] \cdot \left[ D_x \left( e^x + 4 \right) \right] = \left[ 6(e^x + 4)^5 \right] \cdot [e^x] \]

\[ = 6e^x \left( e^x + 4 \right)^5 \] (by Gen. Power Rule)

\[ D_x \left[ \tan(x) \right] = \sec^2(x) \]

\[ D_x \left[ \cot(x) \right] = -\csc^2(x) \]

\[ D_x \left[ \sec(x) \right] = \sec(x) \tan(x) \]

\[ D_x \left[ \csc(x) \right] = -\csc(x) \cot(x) \]

\[ D_x \left[ \sin(4x + 7) \right] = \left[ \cos(4x + 7) \right] \cdot \left[ D_x \left( 4x + 7 \right) \right] = \left[ \cos(4x + 7) \right] \cdot [4] \]

\[ = 4 \cos(4x + 7) \] (by Gen. Trig Rule)

\[ D_x \left( \frac{1}{e^x} \right) = \left[ e^x \right] \cdot \left[ D_x \left( \frac{1}{x} \right) \right] = \left[ e^x \right] \cdot [1] = \frac{1}{e^x} \]
MORE!

\[ D_x\left(5^x\right) = 5^x \ln(5) \]

\[ D_x\left(10^{x^3}\right) = \left[10^{x^3} \ln(10)\right] \cdot \left[D_x\left(x^3\right)\right] = \left[10^{x^3} \ln(10)\right] \cdot \left[3x^2\right] = 3x^2 \cdot 10^{x^3} \ln(10) \]

\[ D_x\left[\ln(7x^2 + 1)\right] = \left[\frac{1}{7x^2 + 1}\right] \cdot \left[D_x\left(7x^2 + 1\right)\right] = \left[\frac{1}{7x^2 + 1}\right] \cdot \left[14x\right] = \frac{14x}{7x^2 + 1} \]

\[ D_x\left[\log_9(x)\right] = D_x\left[\frac{\ln(x)}{\ln(9)}\right] = \left[\frac{1}{\ln(9)}\right] \cdot \left[D_x\left[\ln(x)\right]\right] = \left[\frac{1}{\ln(9)}\right] \cdot \left[\frac{1}{x}\right] = \frac{1}{x \ln(9)} \]

\[ D_x\left[\sin^{-1}(x)\right] = \frac{1}{\sqrt{1-x^2}} \]

\[ D_x\left[\cos^{-1}(x)\right] = -\frac{1}{\sqrt{1-x^2}} \]

\[ D_x\left[\tan^{-1}(x)\right] = \frac{1}{1+x^2} \]

\[ D_x\left[\sec^{-1}(x)\right] = \frac{1}{x \sqrt{x^2 - 1}} \quad \text{(Assume the usual range for } \sec^{-1}(x) \text{ in our class.)} \]

\[ D_x\left[\tan^{-1}(7x)\right] = \left[\frac{1}{1+(7x)^2}\right] \cdot \left[D_x\left(7x\right)\right] = \left[\frac{1}{1+(7x)^2}\right] \cdot \left[7\right] = \frac{7}{1+49x^2} \]

\[ D_x\left[\sinh(x)\right] = \cosh(x) \]

\[ D_x\left[\cosh(x)\right] = \sinh(x) \]

\[ D_x\left[\sech(x)\right] = -\sech(x) \tanh(x) \]
INDEFINITE INTEGRALS (42 POINTS TOTAL)

\[ \int x^9 \, dx = \frac{x^{10}}{10} + C \]

\[ \int \frac{1}{x} \, dx = \ln|x| + C \]

\[ \int e^{-7x} \, dx = \frac{e^{-7x}}{-7} + C = -\frac{e^{-7x}}{7} + C, \text{ or } C - \frac{1}{7e^{7x}} \]

\[ \int 6^x \, dx = \frac{6^x}{\ln(6)} + C \]

\[ \int \sin(x) \, dx = -\cos(x) + C \]

\[ \int \tan(x) \, dx = -\ln|\cos(x)| + C, \text{ or } \ln|\sec(x)| + C \]

\[ \int \cot(x) \, dx = \ln|\sin(x)| + C \]

\[ \int \sec(x) \, dx = \ln|\sec(x) + \tan(x)| + C \]

\[ \int \csc(x) \, dx = \ln|\csc(x) - \cot(x)| + C, \text{ or } -\ln|\csc(x) + \cot(x)| + C \]

\[ \int \cos(3x) \, dx = \frac{1}{3}\sin(3x) + C \]

\[ \int \sec^2(x) \, dx = \tan(x) + C \]

\[ \int \frac{1}{36 + x^2} \, dx = \frac{1}{6}\tan^{-1}\left(\frac{x}{6}\right) + C \]

\[ \int \frac{1}{\sqrt{36 - x^2}} \, dx = \sin^{-1}\left(\frac{x}{6}\right) + C \]

\[ \int \cosh(x) \, dx = \sinh(x) + C \]

WARNING: YOU'VE BEEN DEALING WITH INDEFINITE INTEGRALS. DID YOU FORGET SOMETHING? (+ C)
INVERSE TRIGONOMETRIC FUNCTIONS (6 POINTS TOTAL)

- \( \lim_\limits{x \to \infty} \tan^{-1}(x) = \frac{\pi}{2} \)  
  (Drawing a graph may help.)

- If \( f(x) = \sin^{-1}(x) \), what is the range of \( f \) in interval form (the form with parentheses and/or brackets)? \( \text{Range}(f) = \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \).

HYPERBOLIC FUNCTIONS (6 POINTS TOTAL)

- The definition of \( \sinh(x) \) (as given in class) is: \( \sinh(x) = \frac{e^x - e^{-x}}{2} \)

- Complete the following identity: \( \cosh^2(x) - \sinh^2(x) = 1 \)  
  (We mentioned this identity in class.)

TRIGONOMETRIC IDENTITIES (15 POINTS TOTAL)

Complete each of the following identities, based on the type of identity given.

- \( \tan^2(x) + 1 = \sec^2(x) \)  
  (Pythagorean Identity)

- \( \cos(-x) = \cos(x) \)  
  (Even/Odd Identity)

- \( \sin(2x) = 2\sin(x)\cos(x) \)  
  (Double-Angle Identity)

- \( \cos(2x) = \cos^2(x) - \sin^2(x) \), or \( 1 - 2\sin^2(x) \), or \( 2\cos^2(x) - 1 \)  
  (Double-Angle Identity)  
  (For \( \cos(2x) \), I gave you three versions; you may pick any one.)

- \( \sin^2(x) = \frac{1 - \cos(2x)}{2} \)  
  (Power-Reducing Identity)
SOLUTIONS TO THE FINAL - PART 2
MATH 150 – FALL 2016 – KUNIYUKI
PART 1: 135 POINTS, PART 2: 115 POINTS, TOTAL: 250 POINTS
A scientific calculator and an appropriate sheet of notes are allowed on this final part.

1) Find the following limits. Each answer will be a real number, $\infty$, $-\infty$, or DNE (Does Not Exist). Write $\infty$ or $-\infty$ when appropriate. If a limit does not exist, and $\infty$ and $-\infty$ are inappropriate, write “DNE.” Box in your final answers.
(14 points total)

a) $\lim_{x \to \infty} \frac{\cos(x^3)}{x^4}$

Show all work, as in class. (6 points)

$\lim_{x \to \infty} \frac{\cos(x^3)}{x^4} = 0$. Prove this using the Sandwich / Squeeze Theorem:

$-1 \leq \cos(x^3) \leq 1 \quad (\forall x \in \mathbb{R})$

Observe that $x^4 > 0$, $\forall x > 0$ (we may assume this, since we let $x \to \infty$).

Divide all three parts by $x^4$.

As $x \to \infty$, $-\frac{1}{x^4} \leq \frac{\cos(x^3)}{x^4} \leq \frac{1}{x^4}$ $\implies$ $\lim_{x \to \infty} \frac{\cos(x^3)}{x^4} = 0$.

b) $\lim_{x \to -2^+} \frac{x}{x^2 - 3x - 10}$

Show all work, as in class. (6 points)

$\lim_{x \to -2^+} \frac{x}{x^2 - 3x - 10} = \lim_{x \to -2^+} \frac{x}{(x+2)(x-5)} = \lim_{x \to 0^+} \frac{-2}{0^-} = \infty$

More precisely: $\lim_{x \to -2^+} -\frac{1}{x^2} = 0$, and $\lim_{x \to -2^+} \frac{x}{x^2} = 0$.

Therefore, by the Sandwich / Squeeze Theorem, $\lim_{x \to \infty} \frac{\cos(x^3)}{x^4} = 0$.

c) $\lim_{r \to \infty} \frac{r^3 + 1}{(r^5 + r)^2}$

Answer only is fine. (2 points)

$\lim_{r \to \infty} \frac{r^3 + 1}{(r^5 + r)^2} = 0$, because we are taking a “long-run” limit of a proper (“bottom-heavy”) rational expression as $r \to \infty$. The degree of the denominator (10, since the leading term would be $r^{10}$ after expanding the square) is greater than the degree of the numerator (3).
2) Let \( f(x) = \begin{cases} x + 4, & x \neq 3 \\ 9, & x = 3 \end{cases} \). Classify the discontinuity at \( x = 3 \). Box in one:

(2 points)

<table>
<thead>
<tr>
<th>Infinite discontinuity</th>
<th>Jump discontinuity</th>
<th>Removable discontinuity</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lim_{x \to 3} f(x) = \lim_{x \to 3} (x + 4) = 3 + 4 = 7 \neq 9 ), which is ( f(3) ). (The limit value and the function value exist but are unequal.) If ( f(3) ) were 7, then ( f ) would be continuous at 3.</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

3) Use the limit definition of the derivative to prove that \( D_x (5x^2 - x + 7) = 10x - 1 \), \( \forall x \in \mathbb{R} \). Do not use derivative short cuts we have used in class. (10 points)

Let \( f(x) = 5x^2 - x + 7 \).

\[
\begin{align*}
f''(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\left[5(x+h)^2 - (x+h) + 7\right] - \left[5x^2 - x + 7\right]}{h} \\
&= \lim_{h \to 0} \frac{5(x^2 + 2xh + h^2) - x - h + 7 - 5x^2 + x - 7}{h} \\
&= \lim_{h \to 0} \frac{5x^2 + 10xh + 5h^2 - x - h + 7 - 5x^2 + x - 7}{h} \\
&= \lim_{h \to 0} \frac{10xh + 5h^2 - h}{h} = \lim_{h \to 0} \frac{h(10x + 5h - 1)}{h} = \lim_{h \to 0} (10x + 5h - 1) \\
&= 10x + 5(0) - 1 = 10x - 1. \quad \text{(Q.E.D.)}
\end{align*}
\]

4) Let \( f(x) = \log_4(x) \). Consider the graph of \( y = f(x) \) in the usual xy-plane.

Find a Point-Slope Form of the equation of the tangent line to the graph at the point where \( x = 16 \). Give exact values; you do not have to approximate. (8 points)

\( f(16) = \log_4(16) = 2 \), so the point of interest is \((16, 2)\).

Find \( f'(16) \), the slope \( m \) of the tangent line to the graph at that point.

\[
\begin{align*}
f'(x) &= \frac{d}{dx} \left[ \log_4(x) \right] = \frac{d}{dx} \left[ \frac{\ln(x)}{\ln(4)} \right] \quad \text{(Change-of-Base Property of Logarithms)} \\
&= \left[ \frac{1}{\ln(4)} \right] \cdot \left( \frac{1}{x} \right) = \frac{1}{x \ln(4)} \\
f'(16) &= \frac{1}{16 \ln(4)}, \text{ or } \frac{1}{\ln(4^{16})} \quad \text{(This is our desired slope, } m \text{.)}
\end{align*}
\]

Find a Point-Slope Form of the equation of the tangent line.

\[
y - y_1 = m(x - x_1) \Rightarrow y - 2 = \frac{1}{16 \ln(4)} (x - 16)
\]
5) Assume that \( x \) and \( y \) are differentiable functions of \( t \). Evaluate \( D_t\left(e^{xy}\right) \) when 

\[
x = 3, \quad y = 5, \quad \frac{dx}{dt} = 2, \quad \text{and} \quad \frac{dy}{dt} = -4. \quad (8 \text{ points})
\]

\[
D_t\left(e^{xy}\right) = \left[e^{xy}\cdot D_t(xy)\right] = \left[e^{xy}\cdot \frac{dx}{dt} \cdot y + x \cdot \frac{dy}{dt}\right] \quad \text{(by the Product Rule)}
\]

\[
= \left[e^{(3x)}\cdot [(2)(5) + (3)(-4)]\right] = \left[e^{15}\cdot [10 - 12]\right] = \left[e^{15}\cdot [-2]\right] = -2e^{15}
\]

6) Let \( f(x) = x^3 - 2x^2 - 4x + 1. \) (14 points total)

a) Find the two critical numbers of \( f \).

\[
f'(x) = 3x^2 - 4x - 4 = (3x + 2)(x - 2),
\]

which is never undefined ("DNE") but is 0 at \( x = -\frac{2}{3} \) and \( x = 2 \). These numbers are in \( \text{Dom}(f) \), which is \( \mathbb{R} \), so the critical numbers are: \( -\frac{2}{3} \) and 2.

Note: We could use the Quadratic Formula (QF) with \( a = 3, \ b = -4, \ \text{and} \ c = -4 \).

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(3)(-4)}}{2(3)} = \frac{4 \pm \sqrt{16 + 48}}{6} = \frac{4 \pm 8}{6} = \frac{2 \pm 4}{3} \Rightarrow
\]

\[
x = \frac{2 + 4}{3} = \frac{6}{3} = 2 \quad \text{or} \quad x = \frac{2 - 4}{3} = \frac{-2}{3} = -\frac{2}{3}
\]

b) Consider the graph of \( y = f(x) \), although you do not have to draw it.

Use the First Derivative Test to classify the point at \( x = 2 \) as a local maximum point, a local minimum point, or neither.

\( f' \) is continuous on \( \mathbb{R} \), so the First Derivative Test (1st DT) should apply wherever we have critical numbers (CNs). Both \( f \) and \( f' \) are continuous on \( \mathbb{R} \), so use just the CNs as "fenceposts" on the real number line where \( f' \) could change sign.

<table>
<thead>
<tr>
<th>( f' ) sign (see below)</th>
<th>(-2/3)</th>
<th>Test ( x = 0)</th>
<th>2</th>
<th>Test ( x = 3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f ) sign</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Classify point at CN (1st DT)</td>
<td></td>
<td>\text{L.Min. Pt.}</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[
f'(x) = (3x + 2)(x - 2)
\]

\[
f'(0) = \begin{cases} (+) & (\text{not needed}) \\ (-) & \end{cases} \quad \text{or} \quad \text{Evaluate } f' \text{ at 0 and 3 directly.}
\]

\[
f'(3) = \begin{cases} (+) & \end{cases} \quad \text{or} \quad \text{Evaluate } f' \text{ at 0 and 3 directly.}
\]

Also, the graph of \( y = f'(x) \) is an upward-opening parabola with two distinct \( x \)-intercepts at \( \left(-\frac{2}{3}, 0\right) \) and \( (2, 0) \). The multiplicities of the zeros of \( f' \) are both odd (1), so signs alternate in our "windows." Answer: \text{Local Minimum Point}
c) Use the Second Derivative Test to classify the point at the other critical number as a local maximum point or a local minimum point.

\[ f'(\frac{-2}{3}) = 0, \] so we may apply the Second Derivative Test for \( x = -\frac{2}{3} \).

\[
\begin{align*}
  f'(x) &= 3x^2 - 4x - 4 \quad \Rightarrow \\
  f''(x) &= 6x - 4 \quad \Rightarrow \\
  f''\left(\frac{-2}{3}\right) &= 6\left(\frac{-2}{3}\right) - 4 = -8 \quad \Rightarrow \\
  f''\left(\frac{-2}{3}\right) &< 0
\end{align*}
\]

Therefore, the point at \( x = -\frac{2}{3} \) is a Local Maximum Point.

Think: Concave down \((\cap)\) “at” (actually, on a neighborhood of) \( x = -\frac{2}{3} \).

7) Evaluate the following integrals. (20 points total)

a) \( \int_{9}^{16} \frac{e^{\sqrt{x}}}{\sqrt{x}} \, dx \). Give an exact answer. (10 points)

Let \( u = \sqrt{x} \) or \( x^{\frac{1}{2}} \) \( \Rightarrow \\
\frac{1}{2}x^{-\frac{1}{2}} \, dx \Rightarrow du = \frac{1}{2\sqrt{x}} \, dx \Rightarrow \) (Can use: \( \frac{1}{\sqrt{x}} \, dx = 2 \, du \))

Method 1 (Change the limits of integration.)

\[
\begin{align*}
  x = 9 &\Rightarrow u = \sqrt{9} = 3 \quad \Rightarrow \quad u = 3 \\
  x = 16 &\Rightarrow u = \sqrt{16} = 4 \quad \Rightarrow \quad u = 4
\end{align*}
\]

\[
\begin{align*}
\int_{9}^{16} \frac{e^{\sqrt{x}}}{\sqrt{x}} \, dx &= 2 \int_{9}^{16} \frac{e^{u}}{2u} \, du \quad \text{(by Compensation)} \quad \Rightarrow \\
&= 2 \int_{3}^{4} e^{u} \, du \quad \Rightarrow \quad 2 \left[ e^{u} \right]_{3}^{4} \quad \Rightarrow \quad 2\left( e^{4} - e^{3} \right), \text{ or } 2e^{3}(e-1)
\end{align*}
\]

Method 2 (Work out the corresponding indefinite integral first.)

\[
\begin{align*}
\int \frac{e^{\sqrt{x}}}{\sqrt{x}} \, dx &= 2 \int \frac{e^{\sqrt{x}}}{2\sqrt{x}} \, dx \quad \text{(by Compensation)} \quad \Rightarrow \\
&= 2 \int e^{u} \, du \quad \Rightarrow \quad 2e^{u} + C = 2e^{\sqrt{x}} + C
\end{align*}
\]

Now, apply the FTC directly using our antiderivative (where \( C = 0 \)).

\[
\begin{align*}
\int_{9}^{16} \frac{e^{\sqrt{x}}}{\sqrt{x}} \, dx &= \left[ 2e^{\sqrt{x}} \right]_{9}^{16} = 2 \left[ e^{\sqrt{x}} \right]_{9}^{16} = 2 \left( \left[ e^{4} \right] - \left[ e^{3} \right] \right) \\
&= 2\left( e^{4} - e^{3} \right), \text{ or } 2e^{3}(e-1)
\end{align*}
\]
b) \[ \int \frac{7x}{x^4 + 81} \, dx \] (10 points)

Hint: Consider the Chapter 8 material on inverse trigonometric functions!

Use the template: \( \int \frac{1}{u^2 + 1} \, du \), or \( \int \frac{du}{81 + u^2} = \frac{1}{9} \tan^{-1} \left( \frac{u}{9} \right) + C \).

Let \( u = x^2 \) \( \Rightarrow \) \( u^2 = x^4 \)
\[ du = 2x \, dx \quad \Rightarrow \quad \text{(Can use: } x \, dx = \frac{1}{2} \, du) \]

\[
\int \frac{7x}{x^4 + 81} \, dx = 7 \int \frac{x}{81 + x^4} \, dx = 7 \cdot \frac{1}{2} \int \frac{2x}{81 + x^4} \, dx \quad \text{(by Compensation)}
\]
\[
= \frac{7}{2} \int \frac{du}{81 + u^2} = \frac{7}{2} \left[ \frac{1}{9} \tan^{-1} \left( \frac{u}{9} \right) \right] + C = \left[ \frac{7}{18} \tan^{-1} \left( \frac{x^2}{9} \right) \right] + C
\]

Alternative Method (using basic template \( \int \frac{1}{u^2 + 1} \, du \), or \( \int \frac{du}{1 + u^2} = \tan^{-1} (u) + C \):

\[
\int \frac{7x}{x^4 + 81} \, dx = 7 \int \frac{x}{81 + x^4} \, dx = 7 \int \frac{x}{81 \left( 1 + \frac{x^4}{81} \right)} \, dx = 7 \int \frac{x}{1 + \frac{x^4}{81}} \, dx
\]
\[
= \frac{7}{81} \int \frac{x}{1 + \left( \frac{x^2}{9} \right)^2} \, dx
\]
\[
\left[ \text{Let } u = \frac{x^2}{9}, \text{ or } \frac{1}{9} x^2 \quad \Rightarrow \quad u^2 = \left( \frac{x^2}{9} \right)^2 = \frac{x^4}{81} \right]
\]
\[
\, du = \frac{2}{9} x \, dx \quad \Rightarrow \quad \frac{9}{2} \, du = x \, dx \quad \text{(or use Compensation)}
\]
\[
= \frac{7}{81} \cdot \frac{9}{2} \int_{(9)} \frac{x}{1 + \left( \frac{x^2}{9} \right)^2} \, dx = \frac{7}{18} \int \frac{du}{1 + u^2} = \frac{7}{18} \tan^{-1} (u) + C = \left[ \frac{7}{18} \tan^{-1} \left( \frac{x^2}{9} \right) \right] + C
\]
8) The velocity function for a particle moving along a coordinate line (for \( t > 0 \)) is given by \( v(t) = \frac{1}{t^4} - \sqrt{t} \), where \( t \) is time measured in seconds and velocity is given in meters per second. The particle’s position is measured in meters. Find \( s(t) \), the corresponding position function [rule], if \( s(1) = 2 \) (meters). (9 points)

\[
\int v(t) \, dt = \int \left( \frac{1}{t^4} - \sqrt{t} \right) \, dt = \int \left( t^{-4} - t^{1/2} \right) \, dt \quad \Rightarrow
\]

\[
s(t) = \frac{t^{-3}}{-3} - \frac{t^{3/2}}{3/2} + C = -\frac{1}{3t^3} - \frac{2}{3} t^{3/2} + C
\]

Find \( C \).

We know: \( s(1) = 2 \) (meters).

\[
s(1) = -\frac{1}{3(1)^3} - \frac{2}{3} (1)^{3/2} + C
\]

\[
2 = -\frac{1}{3} - \frac{2}{3} + C
\]

\[
2 = 1 + C
\]

\[
C = 3 \quad \Rightarrow
\]

\[
s(t) = -\frac{1}{3t^3} - \frac{2}{3} t^{3/2} + 3, \text{ or } 3 - \frac{1}{3t^3} - \frac{2}{3} t \sqrt{t}, \text{ or } \frac{9t^3 - 1 - 2t^4 \sqrt{t}}{3t^3},
\]

or \( \frac{9t^3 - 1 - 2t^{9/2}}{3t^3} \) (in meters)

9) The region \( R \) is bounded by the \( x \)-axis, the \( y \)-axis, and the graphs of \( y = \cos(x) \) and \( x = \frac{\pi}{4} \) in the usual \( xy \)-plane. Sketch and shade in the region \( R \). Find the volume of the solid generated by revolving \( R \) about the \( x \)-axis. Evaluate your integral completely. Give an exact answer in simplest form with appropriate units. Distances and lengths are measured in meters. Hint: Use a Power-Reducing Identity. (18 points)

Let \( f(x) = \cos(x) \). Then, \( f \) is nonnegative and continuous on the \( x \)-interval \( \left[ 0, \frac{\pi}{4} \right] \).
The region \( R \) and the equation \( y = \cos(x) \) suggest a “\( dx \) scan” and the Disk Method.

The volume of the solid is given by:

\[
V = \int_0^{\pi/4} \pi (\text{radius})^2 \, dx = \int_0^{\pi/4} \pi \left[ \cos(x) \right]^2 \, dx = \pi \int_0^{\pi/4} \cos^2(x) \, dx
\]

\[
= \pi \int_0^{\pi/4} \frac{1 + \cos(2x)}{2} \, dx \quad \text{(by a Power-Reducing Identity)}
\]

\[
= \frac{\pi}{2} \left[ x + \frac{1}{2} \sin(2x) \right]_0^{\pi/4} \quad \text{(by "Guess-and-check," or using } u = 2x \text{)}
\]

\[
= \frac{\pi}{2} \left[ \left( \frac{\pi}{4} + \frac{1}{2} \sin \left( 2 \cdot \frac{\pi}{4} \right) \right) - \left[ 0 + \frac{1}{2} \sin(2 \cdot 0) \right] \right]
\]

\[
= \frac{\pi}{2} \left[ \left( \frac{\pi}{4} + \frac{1}{2} \sin \left( \frac{\pi}{2} \right) \right) - \left[ 0 + \frac{1}{2} \sin(0) \right] \right]
\]

\[
= \frac{\pi}{2} \left[ \left( \frac{\pi}{4} + 1 \right) \right] - [0] = \frac{\pi}{2} \left( \frac{\pi}{4} + 1 \right) = \frac{\pi}{2} \left( \frac{\pi}{4} + 1 \right) = \frac{\pi}{2} \left( \frac{\pi + 2}{4} \right) = \frac{\pi(\pi + 2)}{8} \text{ m}^3
\]

10) Rewrite \( \tan \left( \sin^{-1} \left( \frac{x}{5} \right) \right) \) as an algebraic expression in \( x \), where \( 0 < x < 5 \).

(7 points)

Let \( \theta = \sin^{-1} \left( \frac{x}{5} \right) \) (\( \theta \) acute) \( \Rightarrow \) \( \sin(\theta) = \frac{x}{5} \)

\[
\tan(\theta) = \frac{\text{opp.}}{\text{adj.}} = \frac{\frac{x}{\sqrt{25 - x^2}}}{1} \]

Note: Rationalizing the denominator is usually unnecessary if the radicand involved is variable.

11) Find \( D_r \left[ \sin^{-1}(\sinh(r)) \right] \). (5 points)

Use the template: \( D_r \left[ \sin^{-1}(u) \right] = \frac{1}{\sqrt{1-u^2}} \cdot D_r(u) \).

\[
D_r \left[ \sin^{-1}(\sinh(r)) \right] = \frac{1}{\sqrt{1 - \sinh^2(r)}} \cdot D_r(\sinh(r)) = \frac{1}{\sqrt{1 - \sinh^2(r)}} \cdot [\cosh(r)]
\]

\[
= \frac{\cosh(r)}{\sqrt{1 - \sinh^2(r)}}. \text{ Note: } 1 - \sinh^2(r) \text{ is not equivalent to } \cosh^2(r), \text{ but } 1 + \sinh^2(r) \text{ is.}
\]