SOLUTIONS TO THE FINAL - PART 1

MATH 150 – SPRING 2017 – KUNIYUKI
PART 1: 135 POINTS, PART 2: 115 POINTS, TOTAL: 250 POINTS

No notes, books, or calculators allowed.

135 points: 45 problems, 3 pts. each. You do not have to algebraically simplify or box in your answers, unless you are instructed to. Fill in all blanks after "=" signs.

DERIVATIVES (66 POINTS TOTAL)

$$D_x\left(x^e\right) = ex^{e-1}$$

$$D_x\left[x^4\sin(x)\right] = \left[D_x\left(x^4\right)\right]\left[\sin(x)\right] + \left[x^4\right]\left(D_x\left[\sin(x)\right]\right) = 4x^3\sin(x) + x^4\cos(x)$$

(By Product Rule of Diff'n.)

$$D_x\left(\frac{x^3}{2x^5+8}\right) = \frac{\left[2x^5+8\right]\cdot\left[D_x\left(x^3\right)\right] - \left[x^3\right]\cdot\left[D_x\left(2x^5+8\right)\right]}{\left(2x^5+8\right)^2} \quad \text{(by Quotient Rule)}$$

$$= \frac{\left[2x^5+8\right]\cdot\left[3x^2\right] - \left[x^3\right]\cdot\left[10x^4\right]}{\left(2x^5+8\right)^2}, \text{ or } \frac{24x^2-4x^7}{\left(2x^5+8\right)^2}, \text{ or } \frac{x^2\left(6-x^5\right)}{\left(x^5+4\right)^2}$$

$$D_x\left(\left[\ln(x)+1\right]^4\right) = 4\left[\ln(x)+1\right]^3\cdot D_x\left[\ln(x)+1\right]$$

$$= 4\left[\ln(x)+1\right]^3\cdot\frac{1}{x}, \text{ or } \frac{4\left[\ln(x)+1\right]^3}{x} \quad \text{(by Gen. Power Rule)}$$

$$D_x\left[\tan(x)\right] = \sec^2(x)$$

$$D_x\left[\cot(x)\right] = -\csc^2(x)$$

$$D_x\left[\sec(x)\right] = \sec(x)\tan(x)$$

$$D_x\left[\csc(x)\right] = -\csc(x)\cot(x)$$

$$D_x\left[\cos(3x-4)\right] = \left[-\sin(3x-4)\right]\cdot\left[D_x(3x-4)\right] = \left[-\sin(3x-4)\right]\cdot\left[3\right]$$

$$= -3\sin(3x-4) \quad \text{(by Generalized Trig Rule)}$$

$$D_x\left(e^{-7x}\right) = \left[e^{-7x}\right]\cdot\left[D_x(-7x)\right] = \left[e^{-7x}\right]\cdot\left[-7\right] = -7e^{-7x}, \text{ or } -\frac{7}{e^{7x}}$$

MORE!

$$D_x\left(9^x\right)= 9^x \ln(9)$$

$$D_x\left(7^{x^4+x}\right)= \left[7^{x^4+x}\ln(7)\right]\cdot\left[D_x\left(x^4+x\right)\right] = \left[7^{x^4+x}\ln(7)\right]\cdot\left[4x^3+1\right]$$

$$D_x\left[\ln(6x+1)\right]= \left[\frac{1}{6x+1}\right]\cdot\left[D_x\left(6x+1\right)\right] = \left[\frac{1}{6x+1}\right]\cdot\left[6\right] = \frac{6}{6x+1}$$

$$D_x\left[\log_3(x)\right]= D_x\left[\frac{\ln(x)}{\ln(3)}\right] = \left[\frac{1}{\ln(3)}\right]\cdot\left(D_x\left[\ln(x)\right]\right) = \left[\frac{1}{\ln(3)}\right]\cdot\left[\frac{1}{x}\right] = \frac{1}{x\ln(3)}$$

$$D_x\left[\sin^{-1}(x)\right]= \frac{1}{\sqrt{1-x^2}}$$

$$D_x\left[\cos^{-1}(x)\right]= -\frac{1}{\sqrt{1-x^2}}$$

$$D_x\left[\tan^{-1}(x)\right]= \frac{1}{1+x^2}$$

$$D_x\left[\sec^{-1}(x)\right]= \frac{1}{x\sqrt{x^2-1}}$$ (Assume the usual range for $\sec^{-1}(x)$ in our class.)

$$D_x\left[\tan^{-1}\left(e^x\right)\right]= \left[\frac{1}{1+\left(e^x\right)^2}\right]\cdot\left[D_x\left(e^x\right)\right] = \left[\frac{1}{1+e^{2x}}\right]\cdot\left[e^x\right] = \frac{e^x}{1+e^{2x}}$$

$$D_x\left[\sinh(x)\right]= \cosh(x)$$

$$D_x\left[\cosh(x)\right]= \sinh(x)$$

$$D_x\left[\operatorname{sech}(x)\right]= -\operatorname{sech}(x)\tanh(x)$$

INDEFINITE INTEGRALS (42 POINTS TOTAL)

$$\int x^5\,dx = \frac{x^6}{6} + C$$

$$\int \frac{1}{x}\,dx = \ln\left|x\right| + C$$

$$\int e^{3x}\,dx = \frac{e^{3x}}{3} + C$$

$$\int 8^x\,dx = \frac{8^x}{\ln(8)} + C$$

$$\int \cos(x)\,dx = \sin(x) + C$$

$$\int \tan(x)\,dx = -\ln\left|\cos(x)\right| + C, \text{ or } \ln\left|\sec(x)\right| + C$$

$$\int \cot(x)\,dx = \ln\left|\sin(x)\right| + C$$

$$\int \sec(x)\,dx = \ln\left|\sec(x) + \tan(x)\right| + C$$

$$\int \csc(x)\,dx = \ln\left|\csc(x) - \cot(x)\right| + C, \text{ or } -\ln\left|\csc(x) + \cot(x)\right| + C$$

$$\int \sin(7x)\,dx = -\frac{1}{7}\cos(7x) + C$$

$$\int \csc(x)\cot(x)\,dx = -\csc(x) + C$$

$$\int \frac{1}{\sqrt{49 - x^2}}\,dx = \sin^{-1}\left(\frac{x}{7}\right) + C$$

$$\int \frac{1}{49 + x^2}\,dx = \frac{1}{7}\tan^{-1}\left(\frac{x}{7}\right) + C$$

$$\int \cosh(x)\,dx = \sinh(x) + C$$

<u>WARNING</u>: YOU'VE BEEN DEALING WITH INDEFINITE INTEGRALS.
DID YOU FORGET SOMETHING? (I'm referring to "$+C$.")

INVERSE TRIGONOMETRIC FUNCTIONS (6 POINTS TOTAL)

• If $f(x) = \cos^{-1}(x)$, what is the range of f in interval form (the form with parentheses and/or brackets)? Range$(f) = \boxed{[0, \pi]}$

• $\lim\limits_{x \to 1^-} \sin^{-1}(x) = \boxed{\frac{\pi}{2}}$ (Drawing a graph may help.)

HYPERBOLIC FUNCTIONS (6 POINTS TOTAL)

• The definition of $\cosh(x)$ (as given in class) is: $\cosh(x) = \boxed{\frac{e^x + e^{-x}}{2}}$

• Complete the following identity: $\cosh^2(x) - \sinh^2(x) = \boxed{1}$
(We mentioned this identity in class.)

TRIGONOMETRIC IDENTITIES (15 POINTS TOTAL)

Complete each of the following identities, based on the type of identity given.

- $1 + \cot^2(x) = \csc^2(x)$ (Pythagorean Identity)
- $\sin(-x) = -\sin(x)$ (Even/Odd Identity)
- $\sin(2x) = 2\sin(x)\cos(x)$ (Double-Angle Identity)
- $\cos(2x) = \cos^2(x) - \sin^2(x)$, or $1 - 2\sin^2(x)$, or $2\cos^2(x) - 1$ (Double-Angle Identity)

(For $\cos(2x)$, I gave you three versions; you may pick any one.)

- $\cos^2(x) = \dfrac{1 + \cos(2x)}{2}$ (Power-Reducing Identity)

SOLUTIONS TO THE FINAL - PART 2

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1) Find the following limits. Each answer will be a real number, ∞, $-\infty$, or DNE (Does Not Exist). Write ∞ or $-\infty$ when appropriate. If a limit does not exist, and ∞ and $-\infty$ are inappropriate, write "DNE." **Box in your final answers.** (16 points total)

a) $\displaystyle\lim_{r\to\infty} \frac{11r^4-7}{8r^4+r^2-1}$ **Answer only is fine.** (2 points)

Answer: $\boxed{\dfrac{11}{8}}$. We take the **ratio of the leading coefficients** of the polynomials in the numerator and the denominator. This is because those polynomials have the **same degree** (4), and we are taking a **"long-run"** limit as $r\to\infty$.

b) $\displaystyle\lim_{t\to-\infty} \frac{t^5+3t^2-7}{t^6-t-1}$ **Answer only is fine.** (2 points)

Answer: $\boxed{0}$, because we are taking a **"long-run"** limit of a **proper ("bottom-heavy")** rational expression as $t\to-\infty$. The degree of the denominator (6) is greater than the degree of the numerator (5).

c) $\displaystyle\lim_{x\to 5^-} \frac{2x+1}{x^2-2x-15}$ **Show all work, as in class.** (6 points)

$$= \lim_{x\to 5^-} \frac{2x+1}{\underbrace{(x-5)}_{\to 0^-}\underbrace{(x+3)}_{\to 8}} \quad \left(\text{Limit Form } \frac{11}{0^-}\right) = \boxed{-\infty}$$

d) $\displaystyle\lim_{x\to 0} \left[x^2\cos\left(\frac{1}{x^3}\right)\right]$ **Show all work, as in class.** (6 points)

Answer: $\boxed{0}$. Prove this using the Sandwich / Squeeze Theorem:

$$-1 \le \cos\left(\frac{1}{x^3}\right) \le 1 \qquad (\forall x \ne 0)$$

Observe that $x^2>0,\ \forall x\ne 0$. Multiply all three parts by x^2.

$$\text{As } x\to 0,\ \underbrace{-x^2}_{\to 0} \le \underbrace{x^2\cos\left(\frac{1}{x^3}\right)}_{\substack{\text{So, }\to 0\\ \text{by the Sandwich/}\\ \text{Squeeze Theorem}}} \le \underbrace{x^2}_{\to 0} \qquad (\forall x\ne 0)$$

More precisely: $\displaystyle\lim_{x\to 0}\left(-x^2\right)=0$, and $\displaystyle\lim_{x\to 0} x^2 = 0$.

Therefore, by the Sandwich / Squeeze Theorem, $\displaystyle\lim_{x\to 0}\left[x^2\cos\left(\frac{1}{x^3}\right)\right]=0$.

2) Use the limit definition of the derivative to prove that $D_x\left(\frac{1}{x}\right) = -\frac{1}{x^2}$ for all real $x \neq 0$. Do **not** use derivative short cuts we have used in class. (11 points)

Let $f(x) = \frac{1}{x}$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{x+h}-\frac{1}{x}}{h} = \lim_{h\to 0}\left(\frac{\left[\frac{1}{x+h}-\frac{1}{x}\right]}{h} \cdot \frac{\left[x(x+h)\right]}{\left[x(x+h)\right]}\right)$$

$$= \lim_{h\to 0} \frac{x-(x+h)}{hx(x+h)} = \lim_{h\to 0} \frac{\cancel{x-x}-h}{hx(x+h)} = \lim_{h\to 0} \frac{\overset{(-1)}{\cancel{-h}}}{\underset{(1)}{\cancel{h}}\,x(x+h)} = \lim_{h\to 0}\left[-\frac{1}{x(x+h)}\right]$$

$$= -\frac{1}{x(x+0)} = -\frac{1}{x^2} \quad \text{(Q.E.D.)}$$

3) Consider the given equation $4y^2 + 3x^4y + 5e^y = 22 + 5e^2$. Assume that it "determines" an implicit differentiable function f such that $y = f(x)$.
Find $\frac{dy}{dx}$ (you may use the y' notation, instead). (12 points)

$$D_x\left(4y^2 + \underbrace{3x^4y}_{\text{Product Rule to } D_x} + 5e^y\right) = D_x\left(22+5e^2\right)$$

$$8yy' + \left[D_x\left(3x^4\right)\right]\cdot\left[y\right] + \left[3x^4\right]\cdot\left[D_x(y)\right] + 5\left[e^y\right]\left[y'\right] = 0$$

$$8yy' + \left[12x^3\right]\cdot\left[y\right] + \left[3x^4\right]\cdot\left[y'\right] + 5\left[e^y\right]\left[y'\right] = 0$$

$$8yy' + 12x^3y + 3x^4y' + 5e^yy' = 0$$

Isolate the terms with y' on one side.

$$8yy' + 3x^4y' + 5e^yy' = -12x^3y$$

Factor out y' on that side.

$$y'\left(8y + 3x^4 + 5e^y\right) = -12x^3y$$

Divide to solve for y'.

$$y' = \boxed{-\frac{12x^3y}{8y+3x^4+5e^y}}$$

4) Consider the graph of the equation in Problem 3), $4y^2+3x^4y+5e^y=22+5e^2$, in the usual xy-plane. Find a Point-Slope Form for the equation of the tangent line to the graph at the point $(-1,2)$. Give an exact answer; do not approximate. You may use your work from Problem 3). (7 points)

$(-1,2)$ satisfies the given equation, so the point $(-1,2)$ lies on the graph of the equation, and we may use the y' formula from Problem 3).

Evaluate y' at $(x,y)=(-1,2)$: $[y']_{(-1,2)} = -\dfrac{12(-1)^3(2)}{8(2)+3(-1)^4+5e^{(2)}} = \boxed{\dfrac{24}{19+5e^2}} \approx 0.429$

Point-Slope Form for the tangent line at $(-1,2)$:

$$y-y_1=m(x-x_1)$$

$$\boxed{y-2=\frac{24}{19+5e^2}(x-(-1)),\ \text{or}\ y-2=\frac{24}{19+5e^2}(x+1)}$$

5) Let $f(x)=x^3+6x^2-7$. Show all work, as in class. (17 points total)

a) Give the x-interval(s) on which f is increasing. Write your answer in interval form. Do not worry about the issue of parentheses vs. brackets. (10 points)

$$f'(x) = 3x^2+12x = 3x(x+4)$$

- f' is never undefined ("DNE"). $f'(x)=0$ at only $x=0$ and $x=-4$. $\text{Dom}(f)=\mathbb{R}$, so 0 and -4 are critical numbers (CNs) in $\text{Dom}(f)$.
- f and f' are everywhere continuous on $\mathbb{R}$, so we use just the CNs as "fenceposts" where f' could change sign.

	Test $x=-5$	**−4**	Test $x=-1$	**0**	Test $x=1$
f' sign	+		−		+
f	↗		↘		↗

$$\begin{aligned} f'(x) &= (3)\ (x)(x+4) \\ f'(-5) &= (+)(-)\ (-) = + \\ f'(-1) &= (+)(-)\ (+) = - \\ f'(1) &= (+)(+)\ (+) = + \end{aligned}$$

- The multiplicities of the zeros of f' are both odd (1), so we get alternating signs in our "windows." Also, the graph of $y=f'(x)$ is an upward-opening parabola with two distinct x-intercepts at $(-4,0)$ and $(0,0)$; this explains the first and last signs.

f is increasing on: $\boxed{(-\infty,-4],\ [0,\infty)}$. (Brackets due to one-sided continuity.)

b) Give the x-interval(s) on which the graph of $y = f(x)$ is concave up. Write your answer in interval form. Do not worry about the issue of parentheses vs. brackets. (7 points)

$$f''(x) = 6x + 12 = 6(x+2)$$

- f'' is never undefined ("DNE"). $f''(x) = 0$ at only $x = -2$.

Dom$(f) = \mathbb{R}$, so -2 is a PIN (Possible Inflection Number) in Dom(f).

- f, f', and f'' are everywhere continuous on $\mathbb{R}$, so we use just the PIN as a "fencepost" where f'' could change sign.

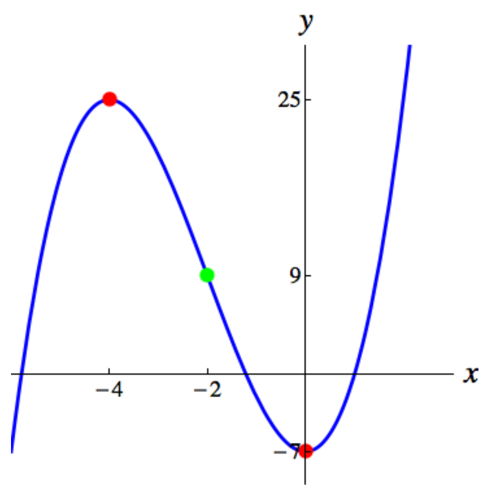
	Test $x = -3$	**-2**	Test $x = 0$
f'' sign	**−**		+
f graph	CD $(\cap)$		CU $(\cup)$

$$\begin{aligned} f''(x) &= (6)(x+2) \\ f''(-3) &= (+)\ (-) = - \\ f''(0) &= (+)\ (+) = + \end{aligned}$$

Also, the graph of $y = f''(x)$ is a rising line with x-intercept at $(-2, 0)$.

The graph of $y = f(x)$ is concave up on: $\boxed{[-2, \infty)}$.

(Bracket due to one-sided continuity.)



6) Evaluate the following integrals. (17 points total)

a) $\int \sin^2(\theta)\, d\theta$ (7 points)

Use a Power-Reducing Identity (PRI).

$$\int \sin^2(\theta)\, d\theta = \int \frac{1-\cos(2\theta)}{2}\, d\theta = \frac{1}{2}\int [1 - \cos(2\theta)]\, d\theta$$

$$= \frac{1}{2}\left[\theta - \frac{1}{2}\sin(2\theta)\right] + C$$ (By "Guess-and-check," or using $u = 2\theta$.)

$$= \boxed{\frac{1}{2}\theta - \frac{1}{4}\sin(2\theta) + C, \text{ or } \frac{2\theta - \sin(2\theta)}{4} + C}$$

b) $\displaystyle\int \frac{x}{\sqrt{25-9x^4}}\,dx$ (10 points)

Hint: Consider the Chapter 8 material on inverse trigonometric functions!

Use the template: $\displaystyle\int \frac{1}{\sqrt{25-u^2}}\,du$, or $\displaystyle\int \frac{du}{\sqrt{25-u^2}} = \sin^{-1}\left(\frac{u}{5}\right)+C$.

$$\left[\begin{array}{llll} \text{Let} \quad u = 3x^2 & \Rightarrow & u^2 = 9x^4 & \\ \quad du = 6x\,dx & \Rightarrow & \frac{1}{6}\,du = x\,dx & (\text{or use Compensation}) \end{array}\right]$$

$$\int \frac{x}{\sqrt{25-9x^4}}\,dx = \frac{1}{6}\int \frac{6x}{\sqrt{25-(3x)^2}}\,dx \ (\text{Compensation}) = \frac{1}{6}\int \frac{du}{\sqrt{25-u^2}}$$

$$= \frac{1}{6}\sin^{-1}\left(\frac{u}{5}\right)+C = \boxed{\frac{1}{6}\sin^{-1}\left(\frac{3x^2}{5}\right)+C}$$

Alternate Method (using more basic template): $\displaystyle\int \frac{1}{\sqrt{1-u^2}}\,du$, or $\displaystyle\int \frac{du}{\sqrt{1-u^2}} = \sin^{-1}(u)+C$

$$\int \frac{x}{\sqrt{25-9x^4}}\,dx = \int \frac{x}{\sqrt{25\left(1-\frac{9x^4}{25}\right)}}\,dx = \frac{1}{5}\int \frac{x}{\sqrt{1-\frac{9x^4}{25}}}\,dx = \frac{1}{5}\int \frac{x}{\sqrt{1-\left(\frac{3x^2}{5}\right)^2}}\,dx$$

$$\left[\begin{array}{llll} \text{Let} \quad u = \frac{3x^2}{5} = \frac{3}{5}x^2 & \Rightarrow & u^2 = \frac{9x^4}{25} & \\ \quad du = \frac{6}{5}x\,dx & \Rightarrow & \frac{5}{6}\,du = x\,dx & (\text{or use Compensation}) \end{array}\right]$$

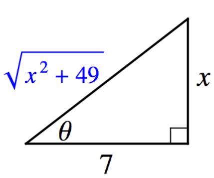
$$= \frac{1}{5}\cdot\frac{5}{6}\int \frac{\frac{6}{5}x}{\sqrt{1-\left(\frac{3x^2}{5}\right)^2}}\,dx \quad (\text{Compensation})$$

$$= \frac{1}{6}\int \frac{du}{\sqrt{1-u^2}} = \frac{1}{6}\sin^{-1}(u)+C = \boxed{\frac{1}{6}\sin^{-1}\left(\frac{3x^2}{5}\right)+C}$$

7) Rewrite $\cos\left(\tan^{-1}\left(\frac{x}{7}\right)\right)$ as an algebraic expression in x. (7 points)

Let $\theta = \tan^{-1}\left(\frac{x}{7}\right) \Rightarrow \tan(\theta) = \frac{x}{7}$, so $\cos\left(\tan^{-1}\left(\frac{x}{7}\right)\right) = \cos(\theta) = \frac{\text{adj.}}{\text{hyp.}} = \boxed{\frac{7}{\sqrt{x^2+49}}}$

Use the Pythagorean Theorem to find the hypotenuse.



(We usually don't rationalize a denominator where a radicand is variable.)

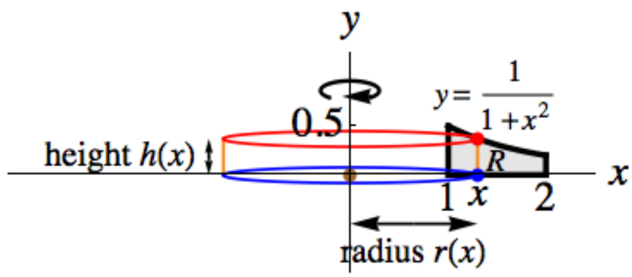
8) Find $D_w\left[\operatorname{sech}^5\left(e^w\right)\right]$. (7 points)

$$D_w\left[\operatorname{sech}^5\left(e^w\right)\right] = D_w\left(\left[\operatorname{sech}\left(e^w\right)\right]^5\right) = 5\left[\operatorname{sech}\left(e^w\right)\right]^4 \cdot D_w\left[\operatorname{sech}\left(e^w\right)\right]$$

$$= 5\left[\operatorname{sech}\left(e^w\right)\right]^4 \cdot \left[-\operatorname{sech}\left(e^w\right)\tanh\left(e^w\right)\right] \cdot \left[D_w\left(e^w\right)\right]$$

$$= 5\left[\operatorname{sech}\left(e^w\right)\right]^4 \cdot \left[-\operatorname{sech}\left(e^w\right)\tanh\left(e^w\right)\right] \cdot \left[e^w\right] = \boxed{-5e^w \operatorname{sech}^5\left(e^w\right)\tanh\left(e^w\right)}$$

9) Distances and lengths are measured in meters. (21 points total)



a) Find the **area** of the shaded region R. **Evaluate** your integral completely. Give an **exact** answer in simplest form with appropriate units, and also **approximate** your answer to four significant digits. (8 points)

The area of the region R is given by: (WARNING: Use **radian** measure!)

$$\int_1^2 \frac{1}{1+x^2}\,dx = \left[\tan^{-1}(x)\right]_1^2 = \left[\tan^{-1}(2)\right] - \left[\tan^{-1}(1)\right] = \boxed{\left[\tan^{-1}(2) - \frac{\pi}{4}\right] \text{m}^2 \approx 0.3218 \text{ m}^2}$$

b) Find the **volume** of the solid generated by revolving the shaded region R about the y-axis. **Evaluate** your integral completely. Give an **exact** answer in simplest form with appropriate units, and also **approximate** your answer to four significant digits. (13 points)

The region R and the given equation (solved for y in terms of x) suggest a "dx scan" and the Cylindrical Shells (Cylinder) Method.

V, the volume of the solid, is given by:

$$V = \int_1^2 2\pi\left[\text{radius } r(x)\right]\left[\text{height } h(x)\right] dx = \int_1^2 2\pi x\left[\frac{1}{1+x^2}\right] dx$$

Let $u = 1 + x^2 \Rightarrow$

$du = 2x\,dx \Rightarrow \left(\text{Can use: } x\,dx = \frac{1}{2}du\right)$

Change the limits of integration:

$x = 1 \Rightarrow u = 1 + (1)^2 = 2 \Rightarrow u = 2$

$x = 2 \Rightarrow u = 1 + (2)^2 = 5 \Rightarrow u = 5$.

$$V = \pi\int_1^2 \left(\frac{1}{1+x^2}\right) \cdot 2x\,dx = \pi\int_2^5 \frac{1}{u}\,du = \pi\left[\ln|u|\right]_2^5$$

$$= \pi\left(\left[\ln|5|\right] - \left[\ln|2|\right]\right) = \boxed{\pi\left[\ln(5) - \ln(2)\right], \text{ or } \pi\ln\left(\frac{5}{2}\right) \text{m}^3 \approx 2.879 \text{ m}^3}$$