SOLUTIONS TO THE FINAL

MATH 151 – SPRING 2004 – KUNIYUKI
105 POINTS TOTAL, BUT 100 POINTS = 100%
An appropriate sheet of notes and a scientific calculator are allowed.

PART 1

45 POINTS TOTAL; 3 POINTS FOR EACH PROBLEM

Give the best answers based on the notes and our discussions in class.

1) We would use a u-substitution to evaluate $\int \sin^4 x \cos^5 x \, dx$. What would be our choice for u?

$$u = \sin x$$

2) We could use a trig substitution to evaluate $\int \frac{x^3}{\sqrt{9x^2+36}} \, dx$. What would we use as our trig substitution?

$$3x = 6\tan\theta, \text{ or}$$
$$x = 2\tan\theta$$

3) We want to find $\int \frac{1}{x^3\left(x^2+25\right)^2} \, dx$ using partial fractions.

Write the form of the partial fraction decomposition for the integrand, $\frac{1}{x^3\left(x^2+25\right)^2}$.

$$\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{Dx+E}{x^2+25} + \frac{Fx+G}{\left(x^2+25\right)^2}$$

4) Find $\lim\limits_{x \to \frac{\pi}{2}^-} \frac{x - \frac{\pi}{2}}{\cos x}$. Write ∞ or $-\infty$ if appropriate. If the limit does not exist, and ∞ and $-\infty$ are inappropriate, write "DNE" (Does Not Exist). Show work!

$$\lim_{x \to \frac{\pi}{2}^-} \frac{x - \frac{\pi}{2}}{\cos x} \left(\frac{0}{0}\right) \overset{\text{L'H}}{=} \lim_{x \to \frac{\pi}{2}^-} \frac{1}{-\sin x}$$
$$= -1$$

5) Which of the following are indeterminate limit forms? Circle all that apply:

$\infty + \infty$ [✓ $\infty - \infty$] [✓ ∞^0]

6) Fill in the boxes (there is more than one possible way):

$$\int_3^\infty \frac{2}{(x-3)^4}\,dx = \int_3^4 \frac{2}{(x-3)^4}\,dx + \int_4^\infty \frac{2}{(x-3)^4}\,dx$$

$$= \lim_{t\to 3^+} \int_t^4 \frac{2}{(x-3)^4}\,dx + \lim_{w\to\infty} \int_4^w \frac{2}{(x-3)^4}\,dx$$

7) For a series $\sum_{n=1}^{\infty} a_n$, we know that the n^{th} partial sum is given by $S_n = \frac{3n^2+1}{2n^2-3}$. What must be the sum of the series?

The sum of the series is given by $\lim_{n\to\infty} S_n$, if it exists.

Since $\frac{3n^2+1}{2n^2-3}$ is a rational expression in which the numerator and the denominator are polynomials of the same degree, we take the ratio of the leading coefficients.

Answer: $\frac{3}{2}$.

8) Consider two positive-term series $\sum_{n=1}^{\infty} c_n$ and $\sum_{n=1}^{\infty} d_n$ such that $c_n \le d_n$ for all $n \ge 1$. According to the Basic Comparison Test, which of the following statements is true? Circle one:

[✓ a) If $\sum_{n=1}^{\infty} d_n$ converges, then $\sum_{n=1}^{\infty} c_n$ converges.]

(If the big brother series converges, so does the little brother series.)

b) If $\sum_{n=1}^{\infty} d_n$ diverges, then $\sum_{n=1}^{\infty} c_n$ diverges.

9) True or False: A positive-term series that converges must be absolutely convergent. Circle one:

[✓ True] False

The corresponding absolute value series must be the original series, since it is positive-term.

10) Write the first four nonzero terms of the Maclaurin series for $f(x) = \tan^{-1} x$.

$$x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7}$$

11) If a power series representation for $f(x)$ is given by $\sum_{n=0}^{\infty} \frac{n^2+1}{3^n} x^n$, find a power series representation for $f'(x)$ using summation notation.

$\sum_{n=1}^{\infty} \frac{n^2+1}{3^n} \cdot nx^{n-1}$ (Start with $n = 1$, because x^{-1} should not be in a power series.)

12) Write the summation notation form for the Taylor series representation of $f(x)$ centered at c, assuming it exists.

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

13) The position of a particle at time t in the usual xy-plane is given by $x = f(t)$, $y = g(t)$ for all t in **R**, where f and g are everywhere differentiable. If, at a particular time, $\frac{dx}{dt} < 0$ and $\frac{dy}{dt} > 0$, what is the direction of the particle at that time? Circle one:

[✓ a) Up and to the left]

b) Up and to the right

c) Down and to the left

d) Down and to the right

14) The graph of the polar equation $r = -3$ is a … (circle one).

[✓ Circle] Line Neither

15) Polar coordinates of a point in a plane are (r, θ). Write the formula we gave in class for the (rectangular or Cartesian) x-coordinate of the point in terms of r and θ.

$x = r\cos\theta$

PART 2

60 POINTS TOTAL

Show all work, simplify as appropriate, and use "good form and procedure" (as in class).
Box in your final answers!

16) Write the formula we gave in class for "work." (3 points)

$$\boxed{W = Fd}$$

17) Solve the separable differential equation $\cos^2 x\, dy - 3y\, dx = 0$. Assume $y > 0$. (12 points)

$$\cos^2 x\, dy = 3y\, dx$$

$$\frac{dy}{y} = \frac{3}{\cos^2 x}\, dx$$

$$\int \frac{dy}{y} = \int \frac{3}{\cos^2 x}\, dx$$

$$\int \frac{dy}{y} = \int 3\sec^2 x\, dx$$

$$\ln\left|y\right| = 3\tan x + C$$

$$\ln y = 3\tan x + C \quad \left(y > 0\right)$$

$$e^{\ln y} = e^{3\tan x + C}$$

$$y = e^{3\tan x} e^{C}$$

$$\boxed{y = Ke^{3\tan x}} \quad \left(K = e^{C}\right)$$

18) Consider the equation $x^2 + 3y^2 - 4x + 6y - 2 = 0$. Its graph is an ellipse in the standard xy-plane. (Show your work at the bottom of the page.) (20 points total)

$$\left(x^2 - 4x\right) + \left(3y^2 + 6y\right) = 2 \qquad \text{(Group terms.)}$$

$$\left(x^2 - 4x\right) + 3\left(y^2 + 2y\right) = 2 \qquad \text{(Factor out leading coefficient[s].)}$$

$$\left(x^2 - 4x + 4\right) + 3\left(y^2 + 2y + 1\right) = 2 + 4 + 3\left(1\right) \qquad \text{(Complete the square and balance the equation.)}$$

$$\left(x - 2\right)^2 + 3\left(y + 1\right)^2 = 9 \qquad \text{(Factor the Perfect Square Trinomials.)}$$

$$\frac{\left(x - 2\right)^2}{9} + \frac{\left(y + 1\right)^2}{3} = 1 \qquad \text{(Divide through by 9; obtain standard form.)}$$

Center: $\left(2,-1\right)$

a^2, which is (by definition) the larger denominator, is underneath the "*x* stuff," so the ellipse is "*x*-long." In other words, its major axis is horizontal.

$$a^2 = 9 \quad \Rightarrow \quad a = 3$$
$$b^2 = 3 \quad \Rightarrow \quad b = \sqrt{3}$$

a) What are the coordinates of the vertices of the ellipse?

$$\left(2 \pm a, -1\right)$$
$$\left(2 \pm 3, -1\right)$$

Vertices: $\boxed{\left(5,-1\right) \text{ and } \left(-1,-1\right)}$.

b) What are the coordinates of the foci of the ellipse?

First, find *c*, the distance between the center and the foci:

$$\begin{aligned} c^2 &= a^2 - b^2 \\ &= 9 - 3 \\ &= 6 \\ c &= \sqrt{6} \end{aligned}$$

The foci are at:

$$\left(2 \pm c, -1\right)$$

$$\boxed{\left(2 \pm \sqrt{6}, -1\right)}$$

19) The graph of $5x^2 + 6\sqrt{3}xy - y^2 - 32 = 0$ is a rotated hyperbola. Use a suitable rotation of axes to find an equation for the graph in an $x'y'$-plane such that the equation has no cross-term. Your final equation must be in standard form for a hyperbola. Also give the angle of rotation. You do <u>not</u> have to graph anything. (25 points total)

Find the angle of rotation, ϕ:

We require that:

$$0^\circ < 2\phi < 180^\circ$$
$$0^\circ < \phi < 90^\circ$$

$$\begin{aligned}\cot(2\phi) &= \frac{A-C}{B}\\ &= \frac{5-(-1)}{6\sqrt{3}}\\ &= \frac{6}{6\sqrt{3}}\\ &= \frac{1}{\sqrt{3}} \quad \text{or} \quad \frac{\sqrt{3}}{3}\end{aligned}$$

$$\Leftrightarrow \tan(2\phi) = \sqrt{3}$$
$$\Leftrightarrow \qquad 2\phi = 60^\circ \quad \left(\text{Given that } 0^\circ < 2\phi < 180^\circ\right)$$
$$\Leftrightarrow \qquad \phi = 30^\circ$$

The angle of rotation is $\mathbf{30^\circ}$.

Find $\sin\phi$ and $\cos\phi$:

$$\sin\phi = \sin 30^\circ = \frac{1}{2}$$
$$\cos\phi = \cos 30^\circ = \frac{\sqrt{3}}{2}$$

Use the rotation of axes formulas:

$$\begin{aligned}\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix}\\ &= \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix}\end{aligned}$$

$$x=\frac{\sqrt{3}}{2}x'-\frac{1}{2}y' \quad\Rightarrow\quad x=\frac{1}{2}\left(\sqrt{3}x'-y'\right)$$

$$y=\frac{1}{2}x'+\frac{\sqrt{3}}{2}y' \quad\Rightarrow\quad y=\frac{1}{2}\left(x'+\sqrt{3}y'\right)$$

Substitute into the given equation:

$$5x^2+6\sqrt{3}xy-y^2-32=0$$

$$5\left[\frac{1}{2}\left(\sqrt{3}x'-y'\right)\right]^2+6\sqrt{3}\left[\frac{1}{2}\left(\sqrt{3}x'-y'\right)\right]\left[\frac{1}{2}\left(x'+\sqrt{3}y'\right)\right]-\left[\frac{1}{2}\left(x'+\sqrt{3}y'\right)\right]^2-32=0$$

For convenience, let’s omit the prime notation for now.

$$5\left[\frac{1}{4}\left(3x^2-2\sqrt{3}xy+y^2\right)\right]+6\sqrt{3}\left[\frac{1}{4}\left(\sqrt{3}x^2+3xy-xy-\sqrt{3}y^2\right)\right]-\left[\frac{1}{4}\left(x^2+2\sqrt{3}xy+3y^2\right)\right]-32=0$$

Let’s multiply through by 4.

$$5\left(3x^2-2\sqrt{3}xy+y^2\right)+6\sqrt{3}\left(\sqrt{3}x^2+2xy-\sqrt{3}y^2\right)-\left(x^2+2\sqrt{3}xy+3y^2\right)-128=0$$

$$15x^2-10\sqrt{3}xy+5y^2+18x^2+12\sqrt{3}xy-18y^2-x^2-2\sqrt{3}xy-3y^2-128=0$$

$$32x^2-16y^2=128$$

Divide through by 128; obtain standard form.

$$\frac{x^2}{4}-\frac{y^2}{8}=1$$

Put in the prime notation.

$$\boxed{\frac{(x')^2}{4}-\frac{(y')^2}{8}=1}$$