QUIZ ON CHAPTER 9
SOLUTIONS

MATH 151 – SPRING 2003 – KUNIYUKI
100 POINTS TOTAL

Evaluate the following integrals.

1) $\int \sin^6 x \cos^5 x \, dx$ (10 points)

The odd power involves $\cos x$, so peel off one $\cos x$ factor.
(We will let u be the "other guy," namely $\sin x$.)

$$= \int \sin^6 x \cos^4 x \cdot \cos x \, dx$$

We then want to rewrite $\cos^4 x$ as a power of $\cos^2 x$ and then use a Pythagorean identity to make more "u"s.

$$= \int \sin^6 x \left(\cos^2 x\right)^2 \cdot \cos x \, dx$$
$$= \int \sin^6 x \left(1 - \sin^2 x\right)^2 \cdot \cos x \, dx$$

Let $u = \sin x$
$du = \cos x \, dx$

$$= \int u^6 \left(1 - u^2\right)^2 du$$
$$= \int u^6 \left(1 - 2u^2 + u^4\right) du$$
$$= \int \left(u^6 - 2u^8 + u^{10}\right) du$$
$$= \frac{u^7}{7} - 2\left(\frac{u^9}{9}\right) + \frac{u^{11}}{11} + C$$

Go back to x!!

$$= \frac{1}{7}\sin^7 x - \frac{2}{9}\sin^9 x + \frac{1}{11}\sin^{11} x + C$$

2) $\int \tan^5 x \sec^3 x \, dx$ (10 points)

We have an odd power of $\tan x$, so peel off a $\sec x \tan x$ factor.
(We will let u be the "other guy," namely $\sec x$.)

$$= \int \tan^4 x \sec^2 x \cdot \sec x \tan x \, dx$$

We then want to rewrite $\tan^4 x$ as a power of $\tan^2 x$ and then use a Pythagorean identity to make more "u"s.

$$= \int \left(\tan^2 x\right)^2 \sec^2 x \cdot \sec x \tan x \, dx$$
$$= \int \left(\sec^2 x - 1\right)^2 \sec^2 x \cdot \sec x \tan x \, dx$$

Let $u = \sec x$
$du = \sec x \tan x \, dx$

$$= \int \left(u^2 - 1\right)^2 u^2 \, du$$
$$= \int \left(u^4 - 2u^2 + 1\right) u^2 \, du$$
$$= \int \left(u^6 - 2u^4 + u^2\right) du$$
$$= \frac{u^7}{7} - 2\left(\frac{u^5}{5}\right) + \frac{u^3}{3} + C$$

Go back to x!!

$$= \frac{1}{7}\sec^7 x - \frac{2}{5}\sec^5 x + \frac{1}{3}\sec^3 x + C$$

3) $\int x^2 \sin x \, dx$ (15 points)

Let's use integration by parts. Both x^2 and $\sin x$ are easy to differentiate and integrate, but we'd rather differentiate x^2.

Let $u = x^2$ Let $dv = \sin x \, dx$
$du = 2x \, dx$ $v = -\cos x$

$$\int x^2 \sin x \, dx = uv - \int v \, du$$
$$= -x^2 \cos x - \int -2x \cos x \, dx$$
$$= -x^2 \cos x + 2\int x \cos x \, dx$$

Use integration by parts again.

Let $u = x$ Let $dv = \cos x \, dx$
$du = dx$ $v = \sin x$

$$\int x \cos x \, dx = uv - \int v \, du$$
$$= x \sin x - \int \sin x \, dx$$
$$= x \sin x + \cos x + C_1$$

Combine our results:

$$\int x^2 \sin x \, dx = -x^2 \cos x + 2\left(x \sin x + \cos x\right) + C$$
$$= -x^2 \cos x + 2x \sin x + 2\cos x + C$$

4) $\displaystyle\int \frac{e^{2x}}{\left(e^x+5\right)^6}\,dx$ (10 points)

Let $u = e^x + 5 \quad \Rightarrow \quad e^x = u - 5$

$du = e^x\ dx$

$$= \int \frac{e^x}{\left(e^x+5\right)^6} \cdot e^x\ dx$$

$$= \int \frac{u-5}{u^6}\ du$$

$$= \int \left(\frac{u}{u^6} - \frac{5}{u^6} \right) du$$

$$= \int \left(u^{-5} - 5u^{-6} \right) du$$

$$= \frac{u^{-4}}{-4} - 5\left(\frac{u^{-5}}{-5} \right) + C$$

$$= -\frac{1}{4}u^{-4} + u^{-5} + C$$

$$= -\frac{1}{4}\left(e^x+5\right)^{-4} + \left(e^x+5\right)^{-5} + C$$

5) $\displaystyle\int \sin^{-1} x\ dx$ (10 points)

Use integration by parts.

Let $u = \sin^{-1} x$ Let $dv = dx$

$du = \dfrac{1}{\sqrt{1-x^2}}\ dx$ $v = x$

$$\int \sin^{-1} x\ dx = uv - \int v\ du$$

$$= x \sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}}\ dx$$

Let $u = 1 - x^2$

$du = -2x\ dx$

$-\dfrac{1}{2}\ du = x\ dx$

$$\int \frac{x}{\sqrt{1-x^2}}\ dx = -\frac{1}{2}\int \frac{du}{\sqrt{u}}$$

$$= -\frac{1}{2}\int u^{-1/2}\ du$$

$$= -\frac{1}{2}\left[\frac{u^{1/2}}{1/2} \right] + C_1$$

$$= -\sqrt{1-x^2} + C_1$$

Combine our results: $\displaystyle\int \sin^{-1} x\ dx = x \sin^{-1} x + \sqrt{1-x^2} + C$

6) $\int \frac{1}{\sqrt{x^2-6x}}\,dx$ (20 points)

Complete the square:

$$x^2-6x=\left(x^2-6x+9\right)-9$$
$$=(x-3)^2-9$$

$$=\int \frac{1}{\sqrt{(x-3)^2-9}}\,dx$$

Trig sub: We have the form $\sqrt{u^2-a^2}$, which leads to the sub $u=a\sec\theta$.

$$x-3=3\sec\theta$$
$$dx=3\sec\theta\tan\theta\,d\theta$$

$$=\int \frac{1}{\underbrace{\sqrt{(3\sec\theta)^2-9}}}\cdot 3\sec\theta\tan\theta\,d\theta$$

$$=\sqrt{9\sec^2\theta-9}$$
$$=\sqrt{9\left(\sec^2\theta-1\right)}$$
$$=3\sqrt{\sec^2\theta-1}$$
$$=3\tan\theta$$

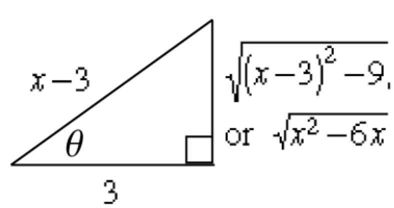
$$=\int \frac{3\sec\theta\tan\theta\,d\theta}{3\tan\theta}$$
$$=\int \sec\theta\,d\theta$$
$$=\ln\left|\sec\theta+\tan\theta\right|+C$$

Find $\sec\theta$:

$$x-3=3\sec\theta$$
$$\sec\theta=\frac{x-3}{3}$$

Find $\tan\theta$:

$$\sec\theta=\frac{x-3}{3}$$



$$\tan\theta=\frac{\sqrt{x^2-6x}}{3}$$

Combine our results:

$$\int \frac{1}{\sqrt{x^2-6x}}\,dx = \ln\left|\sec\theta+\tan\theta\right|+C$$

$$= \ln\left|\frac{x-3}{3}+\frac{\sqrt{x^2-6x}}{3}\right|+C$$

$$\text{or } \ln\left|\frac{x-3+\sqrt{x^2-6x}}{3}\right|+C$$

$$\text{or } \ln\left|x-3+\sqrt{x^2-6x}\right|-\ln 3+C$$

$$\text{or } \ln\left|x-3+\sqrt{x^2-6x}\right|+K$$

7) $\displaystyle\int \frac{x^2+11}{(x-3)\left(x^2+1\right)}\,dx$ (25 points)

Partial Fraction Decomposition Form:

$$\frac{x^2+11}{(x-3)\left(x^2+1\right)} = \frac{A}{x-3}+\frac{Bx+C}{x^2+1}$$

Find A, B, and C:

Multiply both sides by $(x-3)\left(x^2+1\right)$.

$$x^2+11 = A\left(x^2+1\right)+(Bx+C)(x-3) \qquad (\leftarrow \text{Basic equation})$$

Plug in $x = 3$:

$$(3)^2+11 = A\left[(3)^2+1\right]+\left[B(3)+C\right](3-3)$$
$$20 = 10A+0$$
$$A = 2$$

Sub $A = 2$ into the basic equation and expand the right side:

$$x^2+11 = 2\left(x^2+1\right)+(Bx+C)(x-3)$$
$$x^2+11 = 2x^2+2+Bx^2-3Bx+Cx-3C$$
$$x^2+11 = \underbrace{(2+B)}_{=1}x^2+(-3B+C)x+\underbrace{(2-3C)}_{=11}$$

$$2+B=1 \qquad\qquad 2-3C=11$$
$$B=-1 \qquad\qquad -9=3C$$
$$C=-3$$

Partial Fraction Decomposition:

$$\frac{x^2+11}{(x-3)(x^2+1)} = \frac{2}{x-3} + \frac{-x-3}{x^2+1}$$

$$\int \frac{x^2+11}{(x-3)(x^2+1)}\,dx = \int \frac{2}{x-3}\,dx + \int \underbrace{\frac{-x-3}{x^2+1}}_{\text{Split}}\,dx$$

$$= \underbrace{2\int \frac{1}{x-3}\,dx}_{\text{Let } u=x-3} - \underbrace{\int \frac{x}{x^2+1}\,dx}_{\text{Let } u=x^2+1} - 3\int \frac{1}{x^2+1}\,dx$$

Guess-and-Check can help here.

$$= 2\ln|x-3| - \frac{1}{2}\ln\left|\underbrace{x^2+1}_{\geq 0}\right| - 3\tan^{-1}x + C$$

$$= 2\ln|x-3| - \frac{1}{2}\ln\left(x^2+1\right) - 3\tan^{-1}x + C$$