QUIZ ON CHAPTER 10 SOLUTIONS

MATH 151 – SPRING 2003 – KUNIYUKI
100 POINTS TOTAL

1) Find the limits. Write ∞ or $-\infty$ when appropriate. If a limit does not exist, and ∞ and $-\infty$ are inappropriate, write "DNE" (Does Not Exist). Indicate indeterminate forms whenever appropriate, though you don't have to indicate signs for them. (46 points total)

a) $\lim\limits_{x\to 2} \dfrac{x-2\cos(\pi x)}{x^2-4}$ (6 points)

$$\lim_{x\to 2} \frac{x-2\cos(\pi x)}{x^2-4} \quad \frac{\to 2-2\overbrace{\cos(2\pi)}^{=1}=0}{\to 0} \quad \left(\frac{0}{0}\right)$$

$$\overset{\text{L'H}}{=} \lim_{x\to 2} \frac{1-2\left[-\pi\sin(\pi x)\right]}{2x}$$

$$= \lim_{x\to 2} \frac{1+2\pi\sin(\pi x)}{2x} \quad \frac{\to 1+2\pi\overbrace{\sin(2\pi)}^{=0}}{\to 4}$$

$$= \frac{1}{4}$$

b) $\lim\limits_{x\to 0} \dfrac{2^x-1}{3^x-1}$ (4 points)

$$\lim_{x\to 0} \frac{2^x-1}{3^x-1} \quad \frac{\to \overbrace{2^0}^{=1}-1=0}{\to \underbrace{3^0}_{=1}-1=0} \quad \left(\frac{0}{0}\right)$$

$$\overset{\text{L'H}}{=} \lim_{x\to 0} \frac{2^x\ln 2}{3^x\ln 3} \quad \frac{\to \overbrace{2^0}^{=1}\ln 2}{\to \underbrace{3^0}_{=1}\ln 3}$$

$$= \frac{\ln 2}{\ln 3}$$

c) $\lim\limits_{x\to 3} \dfrac{e^{2x}}{\ln(x+1)}$ (2 points)

$$\lim_{x\to 3} \frac{e^{2x}}{\ln(x+1)} \quad \frac{\to e^{2(3)}=e^6}{\to \ln(3+1)=\ln 4}$$

$$= \frac{e^6}{\ln 4}$$

d) $\lim_{x\to\infty} \dfrac{x^2+1}{x\ln x}$ (10 points)

$$\lim_{x\to\infty} \frac{x^2+1}{x\ln x} \quad \frac{\to\infty}{\to\infty} \quad \left(\frac{\infty}{\infty}\right)$$

$$\overset{\text{L'H}}{=} \lim_{x\to\infty} \frac{2x}{(1)(\ln x)+(x)\left(\frac{1}{x}\right)} \quad \underline{\qquad\qquad\qquad\qquad} \leftarrow \text{By Product Rule for } D_x$$

$$= \lim_{x\to\infty} \frac{2x}{\ln x+1} \quad \frac{\to\infty}{\to\infty} \quad \left(\frac{\infty}{\infty}\right)$$

$$\overset{\text{L'H}}{=} \lim_{x\to\infty} \frac{2}{1/x}$$

$$= \lim_{x\to\infty} 2x$$

$$= \infty$$

e) $\lim_{x\to 0^+} x\cot x$ (8 points)

$$\lim_{x\to 0^+} x\cot x \quad (0\cdot\infty)$$

$$= \lim_{x\to 0^+} \frac{x}{\tan x} \quad \frac{\to 0}{\to \tan 0 = 0} \quad \left(\frac{0}{0}\right)$$

$$\overset{\text{L'H}}{=} \lim_{x\to 0^+} \frac{1}{\sec^2 x}$$

$$= \lim_{x\to 0^+} \cos^2 x$$

$$= (\cos 0)^2$$

$$= (1)^2$$

$$= 1$$

f) $\lim_{x\to 0} (\cos x)^{1/x^2}$ (16 points)

$$\lim_{x\to 0} (\cos x)^{1/x^2} \quad (1^\infty)$$

$$\text{Let } y = (\cos x)^{1/x^2}$$

$$\ln y = \ln(\cos x)^{1/x^2}$$

$$= \frac{1}{x^2}\ln\underbrace{(\cos x)}_{\substack{\text{Note: } >0 \\ \text{near } x=0}}$$

$$\lim_{x\to 0} \ln y = \lim_{x\to 0} \frac{1}{x^2}\ln(\cos x)$$

$$= \lim_{x\to 0} \frac{\ln\overbrace{(\cos x)}^{\to 1}}{x^2} \quad \frac{\to 0}{\to 0} \quad \left(\frac{0}{0}\right)$$

$$\overset{\text{L'H}}{=} \lim_{x\to 0} \frac{\frac{1}{\cos x}\cdot(-\sin x)}{2x}$$

$$= \lim_{x\to 0} \frac{-\tan x}{2x} \quad \frac{\to -\tan 0 = 0}{\to 0} \quad \left(\frac{0}{0}\right)$$

$$\overset{\text{L'H}}{=} \lim_{x\to 0} \frac{-\sec^2 x}{2} \quad \frac{\to -(\sec 0)^2 = -1}{\to 2}$$

$$= -\frac{1}{2}$$

Then,

$$\lim_{x\to 0} (\cos x)^{1/x^2} = \lim_{x\to 0} y$$

$$= \lim_{x\to 0} e^{\ln y}$$

$$= e^{-1/2} \quad \text{or} \quad \frac{1}{\sqrt{e}} \quad \text{or} \quad \frac{\sqrt{e}}{e}$$

2) Indicate whether the integral converges or diverges. If it converges, find its value. Either way, show all work, as in class! (51 points total)

a) $\int_{-\infty}^{1} \frac{1}{1+x^2}\, dx$ (9 points)

$$\int_{-\infty}^{1} \underbrace{\frac{1}{1+x^2}}_{\substack{\text{continuous} \\ \text{on } (-\infty,1]}}\, dx = \lim_{t\to -\infty} \int_{t}^{1} \frac{1}{1+x^2}\, dx \quad \text{(if the limit exists)}$$

$$= \lim_{t\to -\infty} \left[\tan^{-1} x\right]_t^1$$

$$= \lim_{t\to -\infty} \left(\left[\tan^{-1}(1)\right] - \left[\tan^{-1} t\right]\right)$$

$$= \frac{\pi}{4} - \left[-\frac{\pi}{2}\right]$$

$$= \frac{\pi}{4} + \frac{\pi}{2}$$

$$= \frac{3\pi}{4}$$

Does the above integral **converge** or diverge?

b) $\int_{-\infty}^{\infty} \frac{x}{\left(x^2+4\right)^{2/3}}\, dx$ (12 points)

$$\int_{-\infty}^{\infty} \underbrace{\frac{x}{\left(x^2+4\right)^{2/3}}}_{\substack{\text{everywhere} \\ \text{continuous}}}\, dx = \int_{-\infty}^{0} \frac{x}{\left(x^2+4\right)^{2/3}}\, dx + \int_{0}^{\infty} \frac{x}{\left(x^2+4\right)^{2/3}}\, dx \quad \text{(if the integrals converge)}$$

Indefinite integral:

$$\int \frac{x}{\left(x^2+4\right)^{2/3}}\,dx$$

Let $\quad u = x^2 + 4$

$$du = 2x\,dx \Rightarrow x\,dx = \frac{1}{2}du$$

$$\begin{aligned}\int \frac{x}{\left(x^2+4\right)^{2/3}}\,dx &= \frac{1}{2}\int \frac{du}{u^{2/3}} \\ &= \frac{1}{2}\int u^{-2/3}\,du \\ &= \frac{1}{2}\left[\frac{u^{1/3}}{1/3}\right]+C \\ &= \frac{3}{2}\left(\sqrt[3]{x^2+4}\right)+C\end{aligned}$$

First integral on the right side:

$$\begin{aligned}\int_{-\infty}^{0} \frac{x}{\left(x^2+4\right)^{2/3}}\,dx &= \lim_{t\to-\infty}\int_{t}^{0} \frac{x}{\left(x^2+4\right)^{2/3}}\,dx \quad \text{(if the limit exists)} \\ &= \lim_{t\to-\infty}\left[\frac{3}{2}\left(\sqrt[3]{x^2+4}\right)\right]_t^0 \\ &= \lim_{t\to-\infty}\left(\underbrace{\left[\frac{3}{2}\left(\sqrt[3]{(0)^2+4}\right)\right]}_{\to\frac{3}{2}\left(\sqrt[3]{4}\right)} - \underbrace{\left[\frac{3}{2}\left(\sqrt[3]{t^2+4}\right)\right]}_{\to\infty}\right) \\ &= -\infty\end{aligned}$$

This integral diverges, so the original integral diverges.

Does the above integral converge or **diverge**?

c) $\displaystyle\int_0^9 \frac{1}{\sqrt[3]{x-1}}\,dx$ (18 points)

$$\underbrace{\int_0^9 \frac{1}{\sqrt[3]{x-1}}}_{\substack{\text{continuous on}\\ [0,1)\cup(1,9]}}\,dx = \int_0^1 \frac{1}{\sqrt[3]{x-1}}\,dx + \int_1^9 \frac{1}{\sqrt[3]{x-1}}\,dx \quad \text{(if the integrals converge)}$$

Indefinite integral:

$$\int \frac{1}{\sqrt[3]{x-1}}\,dx \qquad \text{Let} \quad u = x-1$$
$$du = dx$$

$$\int \frac{1}{\sqrt[3]{u}}\,du = \int u^{-1/3}\,du$$
$$= \frac{u^{2/3}}{2/3} + C$$
$$= \frac{3}{2}(x-1)^{2/3} + C$$

First integral on the right side:

$$\int_0^1 \frac{1}{\sqrt[3]{x-1}}\,dx = \lim_{t\to 1^-} \int_0^t \frac{1}{\sqrt[3]{x-1}}\,dx \quad \text{(if the limit exists)}$$
$$= \lim_{t\to 1^-} \left[\frac{3}{2}(x-1)^{2/3}\right]_0^t$$
$$= \lim_{t\to 1^-} \left(\underbrace{\left[\frac{3}{2}(t-1)^{2/3}\right]}_{\to 0} - \underbrace{\left[\frac{3}{2}(0-1)^{2/3}\right]}_{\to 3/2}\right)$$
$$= -\frac{3}{2}$$

Second integral on the right side:

$$\int_1^9 \frac{1}{\sqrt[3]{x-1}}\,dx = \lim_{t\to 1^+} \int_t^9 \frac{1}{\sqrt[3]{x-1}}\,dx \quad \text{(if the limit exists)}$$
$$= \lim_{t\to 1^+} \left[\frac{3}{2}(x-1)^{2/3}\right]_t^9$$
$$= \lim_{t\to 1^-} \left(\underbrace{\left[\frac{3}{2}(9-1)^{2/3}\right]}_{\to 6} - \underbrace{\left[\frac{3}{2}(t-1)^{2/3}\right]}_{\to 0}\right)$$

Note: $(9-1)^{2/3} = (8)^{2/3} = \left(\sqrt[3]{8}\right)^2 = (2)^2 = 4.$

$$= 6$$

Evaluate the original integral:

$$\int_0^9 \frac{1}{\sqrt[3]{x-1}}\,dx = -\frac{3}{2} + 6$$
$$= \frac{9}{2}$$

Does the above integral **converge** or diverge?

d) $\displaystyle\int_1^\infty \frac{1}{x(\ln x)^2}\,dx$ (12 points)

$$\int_1^\infty \underbrace{\frac{1}{x(\ln x)^2}}_{\substack{\text{continuous on}\\(1,\infty);\text{ undefined at 1}}}\,dx = \int_1^e \frac{1}{x(\ln x)^2}\,dx + \int_e^\infty \frac{1}{x(\ln x)^2}\,dx \quad \text{(if the integrals converge)}$$

Another cut value other than e can be used.

Indefinite integral:

$$\int \frac{1}{x(\ln x)^2}\,dx \qquad \text{Let } u = \ln x, \quad du = \frac{1}{x}\,dx$$

$$= \int \frac{1}{u^2}\,du$$
$$= \int u^{-2}\,du$$
$$= \frac{u^{-1}}{-1} + C$$
$$= -\frac{1}{u} + C$$
$$= -\frac{1}{\ln x} + C$$

First integral on the right side:

$$\int_1^e \frac{1}{x(\ln x)^2}\,dx = \lim_{t\to 1^+} \int_t^e \frac{1}{x(\ln x)^2}\,dx \quad \text{(if the limit exists)}$$
$$= \lim_{t\to 1^+}\left[-\frac{1}{\ln x}\right]_t^e$$
$$= \lim_{t\to 1^+}\left(\underbrace{\left[-\frac{1}{\ln e}\right]}_{\to -1} - \underbrace{\left[-\underbrace{\frac{1}{\underbrace{\ln t}_{\to 0^+}}}_{\to\infty}\right]}_{\to -\infty}\right)$$
$$= \infty$$

This integral diverges, so the original integral diverges.

Does the above integral converge or **diverge**?

3) **True** or False: If the function f is continuous and positive-valued on $[0,\infty)$, and if $\int_0^\infty \frac{1}{f(x)}\,dx$ converges, then $\int_0^\infty \frac{1}{f(x)+x}\,dx$ converges. (3 points)

This is true by the Comparison Test for Integrals.

<u>Note 1:</u>

$$\begin{aligned} &f(x) > 0 \quad \text{on } [0,\infty) \\ \Rightarrow\ &f(x) + x > 0 \quad \text{on } [0,\infty) \end{aligned}$$

<u>Note 2:</u>

$$\begin{aligned} &f(x) + x \ge f(x) \quad \text{on } [0,\infty) \\ \Rightarrow\ &\frac{1}{f(x)+x} \le \frac{1}{f(x)} \quad \text{on } [0,\infty) \end{aligned}$$