QUIZ ON SECTIONS 11.1-5 SOLUTIONS

MATH 151 – SPRING 2003 – KUNIYUKI
102 POINTS TOTAL, BUT 100 POINTS = 100%

1) Find the limits. Write ∞ or $-\infty$ when appropriate. If a limit does not exist, and ∞ and $-\infty$ are inappropriate, write "DNE" (Does Not Exist). You do not have to show work. (9 points total; 3 points each)

a) $\lim_{n\to\infty} a_n$, where $a_n = \left(\frac{n+1}{n}\right)^n$

$$\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$
$$= \boldsymbol{e}$$

b) $\lim_{n\to\infty} a_n$, where $a_n = 6\left(-\frac{2}{5}\right)^n$

$\lim_{n\to\infty} 6\left(-\frac{2}{5}\right)^n = \mathbf{0}$; we have a geometric sequence for which $|r| = \frac{2}{5} < 1$.

c) $\lim_{n\to\infty} a_n$, where $a_n = \frac{\sin^2 n}{\sqrt{n}}$

$\lim_{n\to\infty}\frac{\sin^2 n}{\sqrt{n}} = \mathbf{0}$, by the Squeeze/Sandwich Theorem:

$$0 \le \sin^2 n \le 1$$
$$\underbrace{\frac{0}{\sqrt{n}}}_{\to 0} \le \underbrace{\frac{\sin^2 n}{\sqrt{n}}}_{\substack{\text{So,}\\ \to 0}} \le \underbrace{\frac{1}{\sqrt{n}}}_{\to 0}$$

2) Find the sum of the series $\sum_{n=1}^{\infty}\frac{3^{n+1}}{4^n}$. (8 points)

$$\sum_{n=1}^{\infty}\frac{3^{n+1}}{4^n} = \sum_{n=1}^{\infty}3\left(\frac{3^n}{4^n}\right)$$
$$= \sum_{n=1}^{\infty}3\left(\frac{3}{4}\right)^n$$

Method 1

The first term, a, is $a_1 = 3\left(\frac{3}{4}\right)^1 = \frac{9}{4}$.

The common ratio is $r = \frac{3}{4}$.

$$\text{Sum} = S = \frac{a}{1-r} = \frac{\frac{9}{4}}{1-\frac{3}{4}} = \frac{\frac{9}{4}}{\frac{1}{4}} = \mathbf{9}.$$

Method 2

$$\sum_{n=1}^{\infty} 3\left(\frac{3}{4}\right)^n = \sum_{n=1}^{\infty} 3\left(\frac{3}{4}\right)\left(\frac{3}{4}\right)^{n-1}$$
$$= \sum_{n=1}^{\infty} \frac{9}{4}\left(\frac{3}{4}\right)^{n-1}$$
$$= \sum_{n=1}^{\infty} a\,r^{n-1}, \text{ where } a = \frac{9}{4}, \text{ and } r = \frac{3}{4}$$

$$\text{Again, Sum} = S = \frac{a}{1-r} = \frac{\frac{9}{4}}{1-\frac{3}{4}} = \frac{\frac{9}{4}}{\frac{1}{4}} = \mathbf{9}$$

3) The series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^3}$ is approximated by S_3. According to our discussion in class, what is an upper bound on $|\text{error}|$ for this approximation? (4 points)

$$\sum_{n=1}^{\infty} (-1)^{n-1} \underbrace{\frac{1}{n^3}}_{=a_n} = \underbrace{1 - \frac{1}{8} + \frac{1}{27}}_{=S_3} - \underbrace{\frac{1}{64}}_{=a_4} + \ldots$$

$$|\text{ error }| \le \left|1^{\text{st}} \text{ neglected term}\right| = a_4 = \mathbf{\frac{1}{64}}$$

Note:

$$S \approx 0.9015$$
$$S_3 \approx 0.9120$$

$|\text{ error }| \approx 0.0105$, which is less than [or equal to] $\frac{1}{64} \approx 0.0156$.

4) For each of the following series, box in "Absolutely Convergent," "Conditionally Convergent," or "Divergent," as appropriate. You do <u>not</u> have to show work. (12 points total; 4 points each)

a) $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^{1/3}}$

Absolutely Convergent **Conditionally Convergent** Divergent

You could use the AST to show that this alternating series converges, because $\frac{1}{n^{1/3}}$ decreases (for $n \geq 1$) and approaches 0 as $n \to \infty$.

However, the series $\sum_{n=1}^{\infty}\frac{1}{n^{1/3}}$ diverges, because it is a p-series with $p=\frac{1}{3}\leq 1$.

b) $\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{2}{5}\right)^{n}$

Absolutely Convergent Conditionally Convergent Divergent

The series $\sum_{n=1}^{\infty}\left(\frac{2}{5}\right)^{n}$ converges, because it is a geometric series with $|r|=\frac{2}{5}<1$. Therefore, the given series is absolutely convergent by the ACT.

c) $\sum_{n=1}^{\infty}\left[\frac{1}{n}+(-1)^{n-1}\frac{1}{n}\right]$

Absolutely Convergent Conditionally Convergent **Divergent**

$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$, which is the alternating harmonic series, converges by the AST, but $\sum_{n=1}^{\infty}\frac{1}{n}$, which is the harmonic series (p-series with p = 1), diverges.

Therefore, the original series diverges.

5) True or False: There exists a rearrangement of the terms of the series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$ so that the resulting series converges with sum π.

Box in one: (3 points)

True False

The given series, which is the alternating harmonic series, is conditionally convergent, so for any real number (including π), there exists a rearrangement of terms that yields a series whose sum is that number.

6) For each of the following series:
- Determine whether it converges (write "C") or diverges (write "D").
- State which test you are using (you may abbreviate as in class).
- Show work (as suggested in class).
- Whenever you use the Integral Test, state and verify the assumptions (hypotheses) for the test (as we have done in class). (66 points total)

a) $\sum_{n=2}^{\infty} \frac{n^{3/2}}{n^2 - 3}$ (8 points)

We have a positive-term series.

$\frac{n^{3/2}}{n^2 - 3}$ is an algebraic expression; the Ratio and Root Tests fail and give $L = 1$.

Either the BCT or the LCT will do.

<u>BCT</u>

By using dominant terms in the numerator and denominator, we observe:

$$\frac{n^{3/2}}{n^2} \leq \frac{n^{3/2}}{n^2 - 3} \qquad \text{(for all } n \geq 2\text{)}$$

$$\frac{1}{n^{1/2}} \leq \frac{n^{3/2}}{n^2 - 3} \qquad \text{(for all } n \geq 2\text{)}$$

We know that the "little brother" series $\sum \frac{1}{n^{1/2}}$ diverges, since it is a p-series with $p = \frac{1}{2} \leq 1$. (The fact that the given series starts with $n = 2$ is irrelevant, since that does not affect convergence vs. divergence.) Then, the "big brother" series $\sum_{n=2}^{\infty} \frac{n^{3/2}}{n^2 - 3}$ must also **diverge (D)**.

LCT

Again, use $\sum \frac{1}{n^{1/2}}$ as the comparison series.

$$\lim_{n\to\infty} \frac{\frac{n^{3/.2}}{n^2-3}}{\frac{1}{n^{1/2}}} = \lim_{n\to\infty} \left(\frac{n^{3/.2}}{n^2-3} \cdot n^{1/2} \right)$$

$$= \lim_{n\to\infty} \frac{n^2}{n^2-3} \quad \text{(You may consider dominant terms in num. and den.)}$$

$$= 1, \text{ which is a nonzero real number.}$$

We know that the p-series $\sum \frac{1}{n^{1/2}}$ diverges, so the original series must also **diverge (D)**.

b) $\sum_{n=1}^{\infty} \left[1 + (-1)^n \right]$ (4 points)

The terms alternate between 0 and 2, so they cannot approach 0.
By the n^{th}-Term Test for Divergence, the series **diverges (D)**.

c) $\sum_{n=2}^{\infty} \frac{2n^4+3}{n^7-8}$ (10 points)

We have a positive-term series.

$\frac{2n^4+3}{n^7-8}$ is a rational (in general, algebraic) expression. The Ratio and Root Tests fail; they yield $L = 1$.

Try the LCT. (The BCT does not work immediately.)

Compare $a_n = \frac{2n^4+3}{n^7-8}$ with $b_n = \frac{n^4}{n^7} = \frac{1}{n^3}$. (Consider dominant terms and ignore non-1 constant coefficients.)

(continued…)

$$\lim_{n\to\infty} \frac{\frac{2n^4+3}{n^7-8}}{\frac{1}{n^3}} = \lim_{n\to\infty} \left[\frac{2n^4+3}{n^7-8} \cdot n^3 \right]$$

$$= \lim_{n\to\infty} \frac{2n^7+3n^3}{n^7-8}$$ (You may consider dominant terms in num. and den.)

$= 2$, which is a nonzero real number

We know $\sum \frac{1}{n^3}$ converges, so $\sum_{n=2}^{\infty} \frac{2n^4+3}{n^7-8}$ must also **converge (C)**.

d) $\sum_{n=0}^{\infty} \frac{7^n}{n!}$ (12 points)

We have a positive-term series. The $n!$ hints at the Ratio Test.

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{7^{n+1}}{(n+1)!}}{\frac{7^n}{n!}}$$

$$= \lim_{n\to\infty} \left[\frac{7^{n+1}}{(n+1)!} \cdot \frac{n!}{7^n} \right]$$

$$= \lim_{n\to\infty} \left[\frac{7^{n+1}}{7^n} \cdot \frac{n!}{(n+1)!} \right]$$

$$= \lim_{n\to\infty} 7 \cdot \frac{1}{n+1}$$

$$= 0$$

$$< 1$$

Therefore, the given series **converges (C)**.

e) $\displaystyle\sum_{n=1}^{\infty} n\left(\frac{3}{4}\right)^n$ (12 points)

We have a positive-term series.

Method 1: Ratio Test

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{(n+1)\left(\frac{3}{4}\right)^{n+1}}{n\left(\frac{3}{4}\right)^n}$$
$$=\lim_{n\to\infty}\left[\underbrace{\frac{n+1}{n}}_{\to 1}\cdot\frac{3}{4}\right]$$
$$=\frac{3}{4}$$
$$<1$$

Therefore, the given series **converges (C)**.

Method 2: Root Test

$$\lim_{n\to\infty}\sqrt[n]{a_n}=\lim_{n\to\infty}\sqrt[n]{n\left(\frac{3}{4}\right)^n}$$
$$=\lim_{n\to\infty}\left[\underbrace{\sqrt[n]{n}}_{\to 1}\cdot\underbrace{\sqrt[n]{\left(\frac{3}{4}\right)^n}}_{=\frac{3}{4}}\right]$$
$$=\frac{3}{4}$$
$$<1$$

Therefore, the given series **converges (C)**.

f) $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n\ln n}$ (20 points)

The $\ln n$ and the $\frac{1}{n}$ hint at the Integral Test.

Let $f(x)=\dfrac{1}{x\ln x}$.

Check the hypotheses of the test:

- f is positive-valued on $[2,\infty)$.
- f is continuous on $[2,\infty)$.
- f decreases on $[2,\infty)$. Verify:

$$f(x) = \frac{1}{x\ln x}$$

$$f'(x) = -\frac{\overbrace{D_x(x\ln x)}^{\text{Use Product Rule}}}{(x\ln x)^2}$$

$$= -\frac{(1)(\ln x) + (x)\left(\frac{1}{x}\right)}{(x\ln x)^2}$$

$$= -\frac{\ln x + 1}{(x\ln x)^2} \qquad \text{(Note that } \ln x > 0 \text{ and den.} > 0 \text{ on } [2,\infty).\text{)}$$

$$< 0 \qquad \text{on } [2,\infty)$$

<u>Indefinite integral:</u>

$$\int \frac{1}{x(\ln x)}\,dx \qquad \text{Let } u = \ln x, \quad du = \frac{1}{x}\,dx$$

$$= \int \frac{1}{u}\,du$$

$$= \ln|u| + C$$

$$= \ln|\ln x| + C$$

<u>Now:</u>

$$\int_2^\infty \frac{1}{x(\ln x)}\,dx = \lim_{t\to\infty} \int_2^t \frac{1}{x(\ln x)}\,dx \quad \text{(if the limit exists)}$$

$$= \lim_{t\to\infty} \Big[\ln|\ln x|\Big]_2^t$$

$$= \lim_{t\to\infty} \left(\underbrace{\Big[\ln|\ln t|\Big]}_{\to\infty} - \Big[\ln|\ln 2|\Big] \right)$$

$$= \infty$$

This integral diverges, so the series **diverges (D)**.