QUIZ ON SECTIONS 11.6-8 SOLUTIONS

MATH 151 – SPRING 2003 – KUNIYUKI

PART 1: GRADED OUT OF 80 POINTS; SCORE CUT IN HALF (80 → 40)
PART 2: 60 POINTS
TOTAL ON PARTS 1 AND 2: 100 POINTS

(PART 1)

Fill in the table below. You may use the back for [ungraded] scratch work. Simplify where appropriate, but you do not have to compute factorials.

$f(x)$	First four nonzero terms of the Maclaurin series	Summation notation form for the Maclaurin series	Interval of convergence, I, for the Maclaurin series
$\frac{1}{1-x}$	$1+x+x^2+x^3+\ldots$	$\sum_{n=0}^{\infty} x^n$	$(-1,1)$
$\sin x$	$x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots$	$\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$	$(-\infty,\infty)$
$\cos x$	$1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots$	$\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}$	$(-\infty,\infty)$
$\tan^{-1} x$	$x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\ldots$	$\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$	$[-1,1]$
$\ln(1+x)$	$x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots$	$\sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1}$, or $\sum_{n=1}^{\infty}(-1)^{n-1} \frac{x^n}{n}$	$(-1,1]$
e^x	$1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots$	$\sum_{n=0}^{\infty} \frac{x^n}{n!}$	$(-\infty,\infty)$
$\sinh x$	$x+\frac{x^3}{3!}+\frac{x^5}{5!}+\frac{x^7}{7!}+\ldots$	$\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$	$(-\infty,\infty)$
$\cosh x$	$1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\ldots$	$\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$	$(-\infty,\infty)$

(PART 2)

1) Find the interval of convergence for $\sum_{n=1}^{\infty} \frac{2^n (x-4)^n}{n^2}$. (24 points)

Let $u_n = \frac{2^n (x-4)^n}{n^2}$.

$$
\begin{aligned}
L &= \lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| \\
&= \lim_{n\to\infty} \left| \frac{\frac{2^{n+1}(x-4)^{n+1}}{(n+1)^2}}{\frac{2^n (x-4)^n}{n^2}} \right| \\
&= \lim_{n\to\infty} \left| \frac{2^{n+1}(x-4)^{n+1}}{(n+1)^2} \cdot \frac{n^2}{2^n (x-4)^n} \right| \\
&= \lim_{n\to\infty} \left| \frac{2^{n+1}(x-4)^{n+1}}{2^n (x-4)^n} \cdot \frac{n^2}{(n+1)^2} \right| \\
&= \lim_{n\to\infty} \left| 2(x-4) \cdot \underbrace{\frac{n^2}{(n+1)^2}}_{\to 1} \right| \\
&= \left| 2(x-4) \right|
\end{aligned}
$$

We know the series converges when $L < 1$.

$$
\begin{aligned}
\left| 2(x-4) \right| &< 1 \\
2\left| x-4 \right| &< 1 \\
\left| x-4 \right| &< \frac{1}{2}
\end{aligned}
$$

Solve the absolute value inequality:

$$
\begin{array}{ccccc}
-\frac{1}{2} & < & x-4 & < & \frac{1}{2} \\
-\frac{1}{2}+4 & < & x & < & \frac{1}{2}+4 \\
\frac{7}{2} & < & x & < & \frac{9}{2}
\end{array}
$$

We know that the series converges for these values of x.

Check $x = \frac{7}{2}$:

$$\sum_{n=1}^{\infty} \frac{2^n (x-4)^n}{n^2} = \sum_{n=1}^{\infty} \frac{2^n \left(\frac{7}{2} - 4\right)^n}{n^2}$$

$$= \sum_{n=1}^{\infty} \frac{2^n \left(-\frac{1}{2}\right)^n}{n^2}$$

$$= \sum_{n=1}^{\infty} \frac{2^n \cdot (-1)^n \left(\frac{1}{2^n}\right)}{n^2}$$

$$= \sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2}$$

This series converges by the AST, because it is an alternating series, and $\frac{1}{n^2}$ decreases and approaches 0 as $n \to \infty$.

Check $x = \frac{9}{2}$:

$$\sum_{n=1}^{\infty} \frac{2^n (x-4)^n}{n^2} = \sum_{n=1}^{\infty} \frac{2^n \left(\frac{9}{2} - 4\right)^n}{n^2}$$

$$= \sum_{n=1}^{\infty} \frac{2^n \left(\frac{1}{2}\right)^n}{n^2}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^2}$$

This series converges, because it is a p-series with $p = 2 > 1$.

Answer: $I = \left[\frac{7}{2}, \frac{9}{2}\right]$, or $[3.5, 4.5]$.

2) Evaluate $\int x^2 e^{4x}\,dx$. Hint: The Maclaurin series for e^x is $\sum_{n=0}^{\infty}\frac{x^n}{n!}$.
Just use series; don't use integration by parts. (10 points)

$$e^{4x} = \sum_{n=0}^{\infty}\frac{(4x)^n}{n!} = \sum_{n=0}^{\infty}\frac{4^n x^n}{n!}$$

$$x^2 e^{4x} = \sum_{n=0}^{\infty}\frac{4^n x^{n+2}}{n!} = \sum_{n=0}^{\infty}\frac{4^n}{n!}x^{n+2}$$

$$\int x^2 e^{4x}\,dx = \int \sum_{n=0}^{\infty}\frac{4^n}{n!}x^{n+2}\,dx = \sum_{n=0}^{\infty}\int \frac{4^n}{n!}x^{n+2}\,dx = \sum_{n=0}^{\infty}\frac{4^n}{n!}\cdot\frac{x^{n+3}}{n+3} + C \quad \text{or} \quad \sum_{n=0}^{\infty}\frac{4^n}{n!\,(n+3)}x^{n+3} + C$$

3) Find the first four terms of the Taylor series for $f(x)=\sqrt{x}$ at $c=4$.
(20 points)

$$f(x) = x^{1/2}$$
$$f'(x) = \frac{1}{2}x^{-1/2}$$
$$f''(x) = -\frac{1}{4}x^{-3/2}$$
$$f'''(x) = \frac{3}{8}x^{-5/2}$$

$$f(4) = (4)^{1/2} = 2$$
$$f'(4) = \frac{1}{2}(4)^{-1/2} = \frac{1}{2}\cdot\frac{1}{\sqrt{4}} = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$$
$$f''(4) = -\frac{1}{4}(4)^{-3/2} = -\frac{1}{4}\cdot\frac{1}{\left(\sqrt{4}\right)^3} = -\frac{1}{4}\cdot\frac{1}{8} = -\frac{1}{32}$$
$$f'''(4) = \frac{3}{8}(4)^{-5/2} = \frac{3}{8}\cdot\frac{1}{\left(\sqrt{4}\right)^5} = \frac{3}{8}\cdot\frac{1}{32} = \frac{3}{256}$$

Taylor series at $c=4$:

$$f(4) + f'(4)(x-4) + \frac{f''(4)}{2!}(x-4)^2 + \frac{f'''(4)}{3!}(x-4)^3 + \ldots$$
$$= 2 + \frac{1}{4}(x-4) + \frac{-1/32}{2}(x-4)^2 + \frac{3/256}{6}(x-4)^3 + \ldots$$
$$= 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 + \frac{1}{512}(x-4)^3 + \ldots$$

Note: If you plug in $x=5$, you get $\sqrt{5}$ accurate to three decimal places: 2.236.

TRUE/FALSE

Circle "True" or "False" as appropriate. (6 points total; 3 points each)

a) If $f, f', f'', \ldots$ all exist everywhere (i.e., for all values of x, say), then the Taylor series at any real number c is a valid representation for f everywhere.

True **False**

See Example 6 on pp.586-7. $f, f', f'', \ldots$ all exist everywhere, but the series is only valid for $x = 0$. (We say that there is "no Taylor series representation for *f*.") In order for the Taylor series to be valid everywhere, we also require that the remainder term $R_n(x) \to 0$ for all real x.

b) The Maclaurin series for $f(x) = \cos x$ is a valid representation for f everywhere.

True False

See Example 2 on p.584. The analysis for $f(x) = \cos x$ is similar to the analysis for $f(x) = \sin x$.