QUIZ ON CHAPTER 13 SOLUTIONS

MATH 151 – SPRING 2003 – KUNIYUKI
100 POINTS TOTAL

When graphing, be reasonably accurate, and clearly indicate orientation. Use as many arrowheads as appropriate. Clearly indicate *x*- and *y*-intercepts, endpoints, and extreme points when feasible.

1) Find a rectangular equation for the curve described by:

$$x = t^2 + 3$$
$$y = 4 - t$$
$$t \text{ in } \mathbf{R}$$

(4 points)

$$y = 4 - t \quad \Rightarrow \quad t = 4 - y$$

Together with $x = t^2 + 3 \quad \Rightarrow \quad \boldsymbol{x = (4 - y)^2 + 3}$

or $\boldsymbol{x = y^2 - 8y + 19}$

2) Find a rectangular equation in *x* and *y* that has the same graph as the polar equation $r^2 = 6\sec\theta\csc\theta$. (6 points)

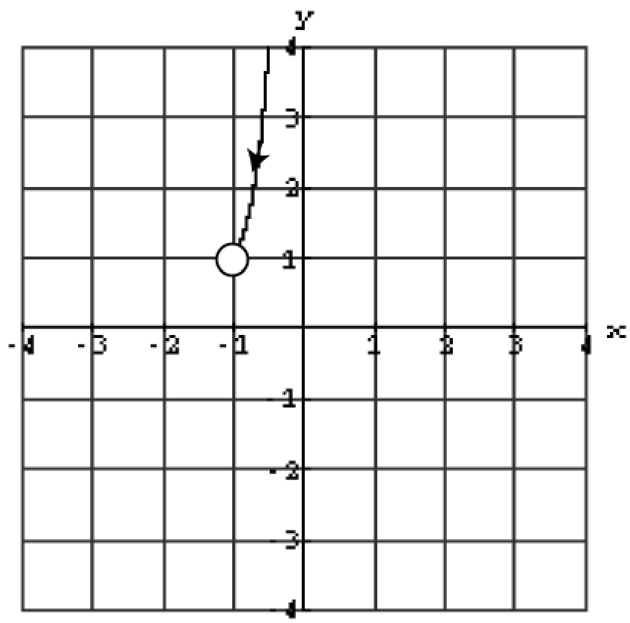
$$r^2 = 6\sec\theta\csc\theta$$
$$r^2 = 6 \cdot \frac{1}{\cos\theta} \cdot \frac{1}{\sin\theta}$$
$$r^2\cos\theta\sin\theta = 6$$
$$(r\cos\theta)(r\sin\theta) = 6$$
$$\boldsymbol{xy = 6 \quad \text{or} \quad y = \frac{6}{x}}$$

Note: The graph of this is a hyperbola.

3) Sketch the graph of C using the grid below, where C is described by:

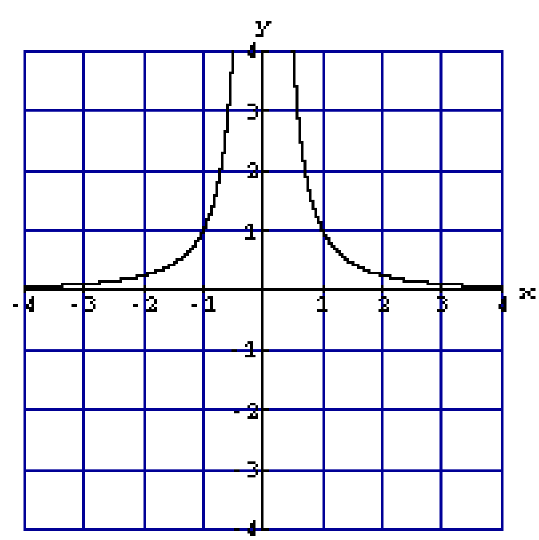
$$x = \cos t$$
$$y = \sec^2 t$$
$$\frac{\pi}{2} < t < \pi$$

(10 points)



Clues:

Because $\sec^2 x = \frac{1}{\cos^2 x}$, the rectangular equation is $y = \frac{1}{x^2}$ (note that we have an even function of x here). Its unrestricted graph is:



As t increases from $\frac{\pi}{2}$ to π (excluding the endpoints themselves),

$\cos t$ stays negative; in particular, it decreases from 0 to –1 (excluding 0 and –1, themselves).

4) Consider the curve described by:

$$x = e^{2t}$$
$$y = \sqrt{t}$$
$$t \geq 0$$

(16 points total)

a) Find the slope of the tangent line at the point on the curve that corresponds to $t = 4$. Give an exact answer; don't approximate. (10 points)

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dy/dt}{dx/dt} \\
&= \frac{\frac{1}{2}t^{-1/2}}{2e^{2t}} \\
&= \frac{\frac{1}{2\sqrt{t}}}{2e^{2t}} \\
&= \frac{1}{4e^{2t}\sqrt{t}} \\
\left[\frac{dy}{dx}\right]_{t=4} &= \frac{1}{4e^{2(4)}\sqrt{4}} \\
&= \mathbf{\frac{1}{8e^{8}}}
\end{aligned}$$

b) Set up, **but do not evaluate**, an integral that represents the length of the curve from the point corresponding to $t = 1$ to the point corresponding to $t = 4$. (6 points)

$$\int_1^4 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \quad = \quad \mathbf{\int_1^4 \sqrt{\left(2e^{2t}\right)^2 + \left(\frac{1}{2\sqrt{t}}\right)^2}\, dt}$$

5) Consider the curve described by:

$$x = t^3 + 1$$
$$y = \sin t$$
$$t \text{ in } \mathbf{R}$$

Find $\frac{d^2y}{dx^2}$ in terms of t. (10 points)

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\cos t}{3t^2}$$

$$\frac{d^2y}{dx^2} = \frac{D_t\left(\frac{\cos t}{3t^2}\right)}{dx/dt}$$

Use the Quotient Rule to get the numerator.

$$= \frac{\frac{(3t^2)(-\sin t) - (\cos t)(6t)}{(3t^2)^2}}{3t^2}$$

$$= \frac{\frac{-3t^2 \sin t - 6t \cos t}{9t^4}}{3t^2}$$

$$= \frac{-3t^2 \sin t - 6t \cos t}{27t^6}$$

$$= \frac{-3t(t \sin t + 2\cos t)}{27t^6}$$

$$= -\frac{\mathbf{t \sin t + 2\cos t}}{\mathbf{9t^5}}$$

6) Find the slope of the tangent line to the graph of the polar equation $r = 2 - 3\sin\theta$ at the point corresponding to $\theta = \frac{\pi}{6}$. Give an exact answer, but you do <u>not</u> have to rationalize the denominator. (18 points)

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

$$= \frac{\frac{d}{d\theta}(r \sin\theta)}{\frac{d}{d\theta}(r \cos\theta)} \quad \begin{array}{l} \leftarrow \text{Use Product Rule} \\ \leftarrow \text{Use Product Rule} \end{array}$$

$$= \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

$$r = 2 - 3\sin\theta$$

$$\frac{dr}{d\theta} = -3\cos\theta$$

$$\left[\frac{dr}{d\theta}\right]_{\theta=\frac{\pi}{6}} = -3\cos\frac{\pi}{6}$$

$$= -3\cdot\frac{\sqrt{3}}{2}$$

$$= -\frac{3\sqrt{3}}{2}$$

We will plug in $\left[\frac{dr}{d\theta}\right]_{\theta=\frac{\pi}{6}} = -\frac{3\sqrt{3}}{2}$.

We will also plug in:

$$\left(\theta = \frac{\pi}{6}\right) \Rightarrow \quad \sin\theta = \sin\frac{\pi}{6} = \frac{1}{2},$$

$$\cos\theta = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2},$$

$$r = 2 - 3\sin\theta = 2 - 3\sin\frac{\pi}{6} = 2 - 3\left(\frac{1}{2}\right) = \frac{1}{2}$$

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

$$= \frac{\left(-\frac{3\sqrt{3}}{2}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)}{\left(-\frac{3\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)}$$

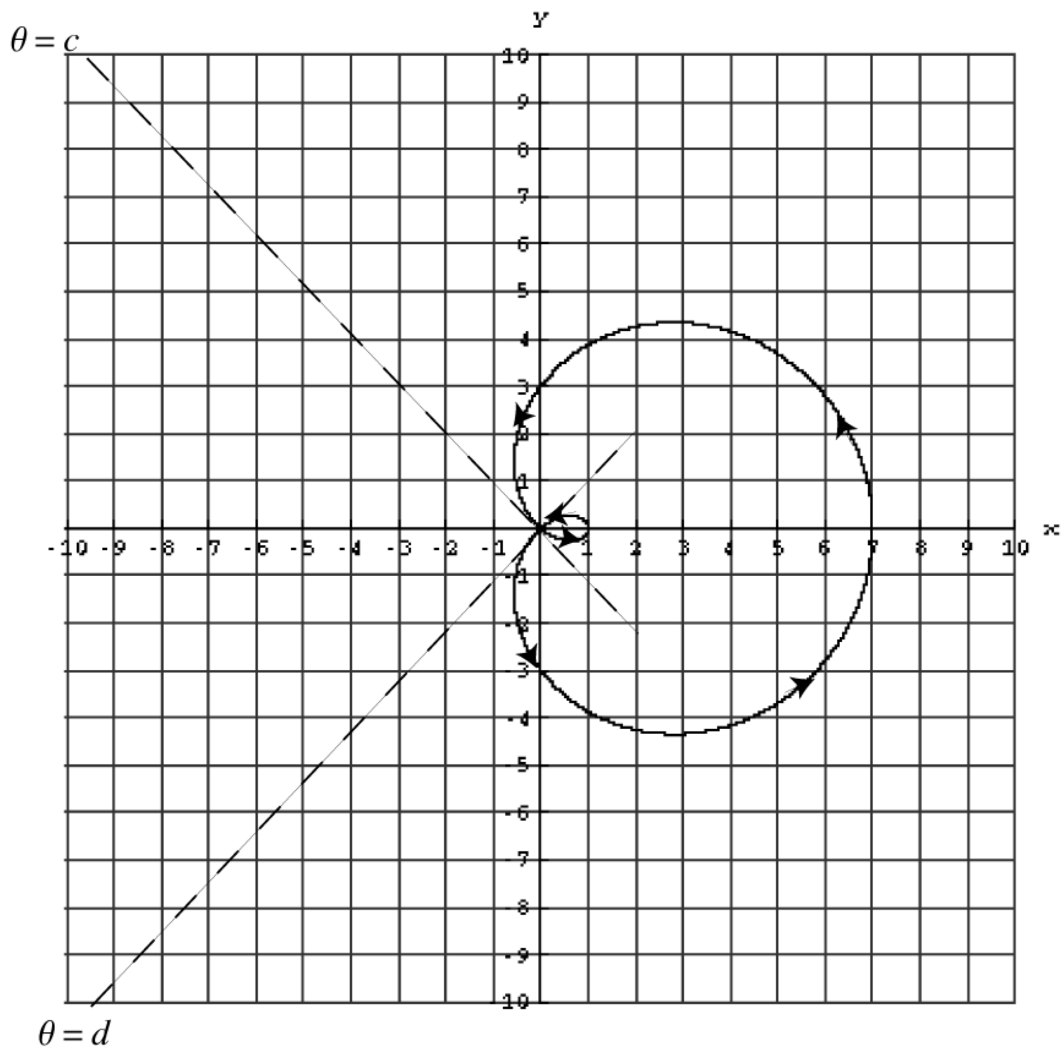
$$= \frac{-\frac{3\sqrt{3}}{4} + \frac{\sqrt{3}}{4}}{-\frac{9}{4} - \frac{1}{4}}$$

$$= \frac{-\frac{2\sqrt{3}}{4}}{-\frac{10}{4}}$$

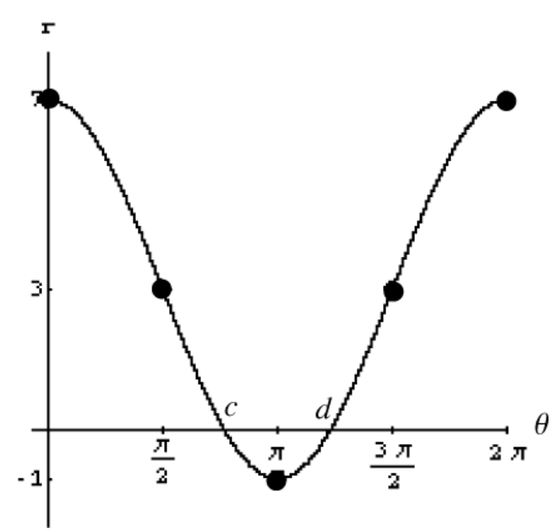
$$= \frac{2\sqrt{3}}{10}$$

$$= \boldsymbol{\frac{\sqrt{3}}{5}}$$

7) Sketch the graph of $r = 3 + 4\cos\theta$ using the grid below. You do <u>not</u> have to determine the exact value(s) of θ for which $r = 0$. (20 points)

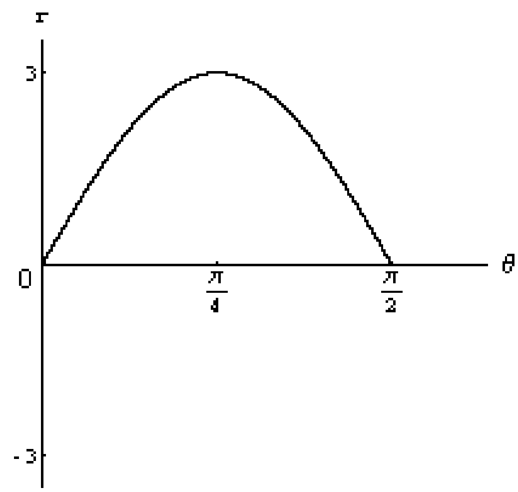


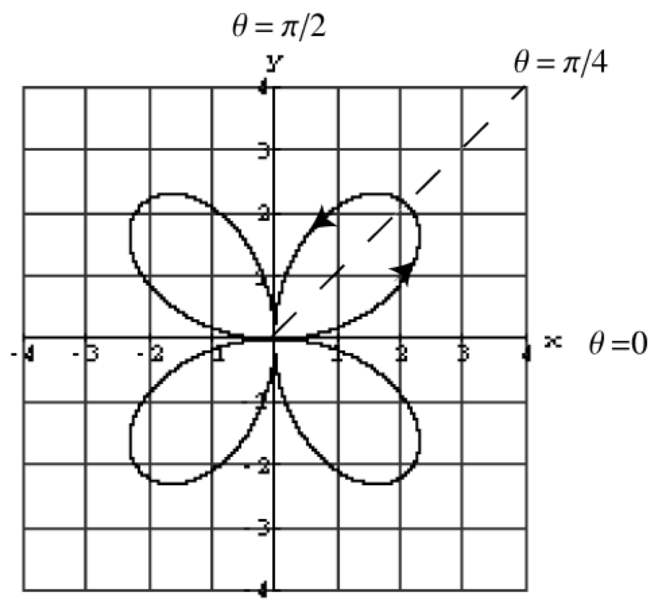
It helped to graph r against θ in Cartesian coordinates:



8) Find the area of the region bounded by one loop of the graph of the polar equation $r = 3\sin(2\theta)$. You may use the grid below as a guide. (16 points)

It turns out we just need to graph r against θ in Cartesian coordinates from $\theta = 0$ to $\theta = \frac{\pi}{2}$.





One loop [of this four-leafed rose] is traced out from $\theta = 0$ to $\theta = \frac{\pi}{2}$, but we can exploit symmetry.

$$
\begin{aligned}
A &= 2\int_0^{\pi/4} \frac{1}{2} r^2 \, d\theta \\
&= \int_0^{\pi/4} r^2 \, d\theta \\
&= \int_0^{\pi/4} \left[3\sin(2\theta)\right]^2 \, d\theta \\
&= \int_0^{\pi/4} 9\sin^2(2\theta) \, d\theta \\
&= 9\int_0^{\pi/4} \frac{1-\cos(4\theta)}{2} \, d\theta \\
&= \frac{9}{2}\left[\theta - \frac{1}{4}\sin(4\theta)\right]_0^{\pi/4} \\
&= \frac{9}{2}\left(\left[\frac{\pi}{4} - \frac{1}{4}\sin(\pi)\right] - \left[0 - \frac{1}{4}\sin(0)\right]\right) \\
&= \frac{9}{2}\left(\left[\frac{\pi}{4} - 0\right] - [0]\right) \\
&= \boldsymbol{\frac{9\pi}{8}}
\end{aligned}
$$