QUIZ ON CHAPTER 12 SOLUTIONS

MATH 151 – SPRING 2003 – KUNIYUKI
100 POINTS TOTAL

You may use mixed numbers instead of improper fractions in your answers.
Don't approximate.

1) Consider the equation $x+6y^2-48y+93=0$. Its graph is a parabola in the standard xy-plane. (Show your work at the bottom of the page.) (13 points total)

   a) Which way does the parabola open (down, left, right, or up)?

$$x+6y^2-48y+93=0$$
$$x=\underbrace{-6y^2+48y-93}_{f(y)}$$

The equation expresses x as a function of y, so the parabola either opens right or left. The leading coefficient of $f(y)$ is negative $(a=-6<0)$, so the parabola opens **left**.

   b) What are the coordinates of the vertex of the parabola?

$$\begin{aligned} k &= y\text{ - coordinate of vertex} \\ &= -\frac{b}{2a} \\ &= -\frac{48}{2(-6)} \\ &= 4 \end{aligned}$$

$$\begin{aligned} h &= x\text{ - coordinate of vertex} \\ &= f(4) \\ &= -6(4)^2+48(4)-93 \\ &= 3 \end{aligned}$$

Vertex: $\mathbf{(3,4)}$

c) What are the coordinates of the focus of the parabola?

Find p, the signed distance from the vertex to the focus.

$$\begin{aligned} p &= \frac{1}{4a} \\ &= \frac{1}{4(-6)} \\ &= -\frac{1}{24} \end{aligned}$$

The parabola opens left, so the focus lies to the left of the vertex. (Also, $p < 0$.)

The vertex is at $(3,4)$, so the focus lies at:

$$\left(3-\frac{1}{24},4\right)$$

$$\mathbf{\left(2\frac{23}{24},4\right)}$$

2) Consider the equation $3x^2+2y^2+18x-4y+17=0$. Its graph is an ellipse in the standard *xy*-plane. (Show your work at the bottom of the page.) (24 points total)

$$3x^2+2y^2+18x-4y+17=0$$

Group terms.

$$\left(3x^2+18x\right)+\left(2y^2-4y\right)=-17$$

Factor out the leading coefficients.

$$3\left(x^2+6x\right)+2\left(y^2-2y\right)=-17$$

Complete the square in the two groups.

$$3\left(x^2+6x+9\right)+2\left(y^2-2y+1\right)=-17+3(9)+2(1)$$

Factor the Perfect Square Trinomials.

$$3(x+3)^2+2(y-1)^2=12$$

$$\frac{3(x+3)^2}{12}+\frac{2(y-1)^2}{12}=\frac{12}{12}$$

$$\frac{(x+3)^2}{4}+\frac{(y-1)^2}{6}=1$$

The center is $(-3,1)$, which makes the left side equal to 0.

Since $6 > 4$, we have a y-long ellipse, and

$$a^2 = 6 \qquad\qquad b^2 = 4$$
$$a = \sqrt{6} \qquad\qquad b = 2 \quad (\text{Not needed in this problem.})$$

Find c:

$$c^2 = a^2 - b^2$$
$$= 6 - 4$$
$$= 2$$
$$c = \sqrt{2}$$

a) What are the coordinates of the vertices of the ellipse?

Move up and down $a = \sqrt{6}$ units from the center to get the vertices.
The center is $(-3, 1)$. The vertices are:

$$\left(\mathbf{-3, 1 \pm \sqrt{6}}\right)$$

b) What are the coordinates of the foci of the ellipse?

Move up and down $c = \sqrt{2}$ units from the center to get the foci.
The center is $(-3, 1)$. The foci are:

$$\left(\mathbf{-3, 1 \pm \sqrt{2}}\right)$$

c) What is the eccentricity of the ellipse?

$$e = \frac{c}{a} = \frac{\sqrt{2}}{\sqrt{6}} = \sqrt{\frac{2}{6}} = \sqrt{\frac{1}{3}} = \frac{\sqrt{\mathbf{3}}}{\mathbf{3}}$$

3) Consider the equation $\frac{(y-3)^2}{9} - \frac{(x+2)^2}{25} = 1$. Its graph is a hyperbola in the standard xy-plane. (17 points total)

Preliminaries:

The center is $(h, k) = (-2, 3)$. (This makes the left side equal to 0.)

$$a^2 = 9 \quad (\text{"on the left"}) \qquad\qquad b^2 = 25$$
$$a = 3 \quad (\text{Take the positive root.}) \qquad b = 5 \quad (\text{Take the positive root.})$$

Since the "y stuff" is on the left of the "$-$" in the equation, the hyperbola opens up and down.

a) What are the coordinates of the foci of the hyperbola?

Find c:

$$c^2 = a^2 + b^2$$
$$= 9 + 25$$
$$= 34$$

$$c = \sqrt{34}$$

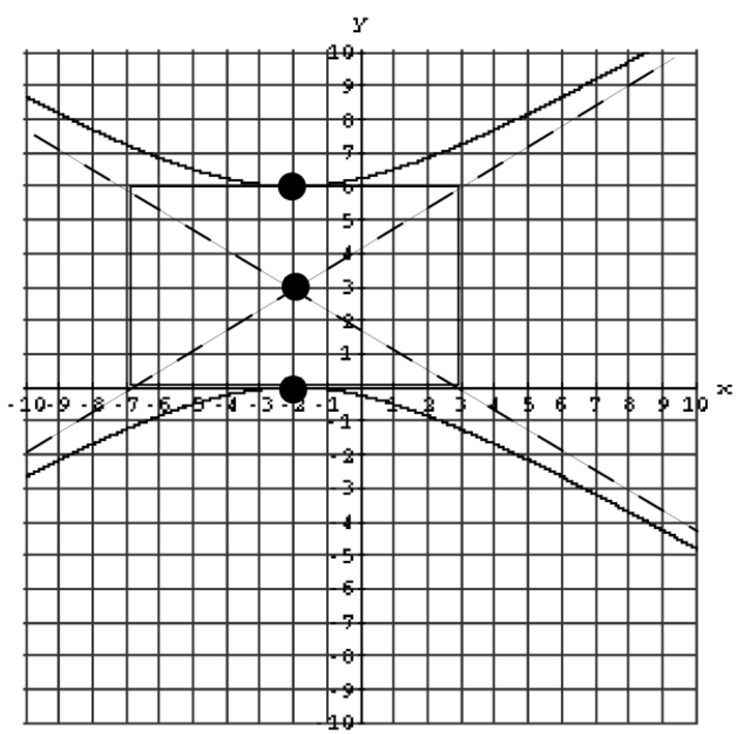
Therefore, the foci must lie directly above and below the center.
The foci are $\left(\mathbf{-2, 3 \pm \sqrt{34}}\right)$.

b) What are the equations of the asymptotes of the hyperbola?

The hyperbola opens up and down.

$$y - k = \pm\frac{a}{b}(x - h)$$
$$y - 3 = \pm\frac{3}{5}(x - (-2))$$
$$\boldsymbol{y - 3 = \pm\frac{3}{5}(x + 2)}$$

c) Sketch the graph of the hyperbola using the coordinate grid below. Make sure you accurately plot/graph the center, the vertices, and the asymptotes of the hyperbola.



4) Find an equation of the parabola that has vertex $(7,-2)$ and focus $(7,1)$.
(8 points)

Since the focus is directly above the vertex, the parabola opens up, and x is the variable that is ultimately squared.

Form of equation:

$$\begin{aligned} y-k &= a(x-h)^2 \\ y-(-2) &= a(x-7)^2 \\ y+2 &= a(x-7)^2 \end{aligned}$$

Find a:

The focus is 3 units directly above the vertex, so $p=3$.

$$\begin{aligned} a &= \frac{1}{4p} \\ &= \frac{1}{4(3)} \\ &= \frac{1}{12} \end{aligned}$$

Equation:

$$\boldsymbol{y+2=\frac{1}{12}(x-7)^2}$$

5) Find an equation of the hyperbola that has vertices $(\pm 4,0)$ and foci $(\pm 7,0)$.
(10 points)

The center must be (0,0), since it is the midpoint of the line segment connecting the two vertices.

The center, vertices, and foci all lie along the x-axis, so the hyperbola opens left and right.

Form of equation:

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

Each vertex is 4 units away from the center, so $a=4 \Rightarrow a^2=16$.

Each focus is 7 units away from the center, so $c=7 \Rightarrow c^2=49$.

Find b^2:

$$c^2 = a^2 + b^2$$
$$49 = 16 + b^2$$
$$b^2 = 33$$

Equation:

$$\boldsymbol{\frac{x^2}{16} - \frac{y^2}{33} = 1}$$

6) A particular rotated conic section has equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, for particular values of $A, B, \ldots F$. If $B^2 - 4AC = -6$, what type of conic do we have – an ellipse, a hyperbola, or a parabola? (3 points)

Since the discriminant is negative, the conic is an **ellipse**.

7) The graph of $5x^2 - 2\sqrt{3}xy + 7y^2 - 24 = 0$ is a rotated ellipse. Use a suitable rotation of axes to find an equation for the graph in an $x'y'$-plane such that the equation has no cross-term. Your final equation must be in standard form for an ellipse. Also give the angle of rotation. You do <u>not</u> have to graph anything. (25 points total)

Find the angle of rotation, ϕ:

We require that:

$0^\circ < 2\phi < 180^\circ$, and
$0^\circ < \phi < 90^\circ$.

$$\begin{aligned}\cot(2\phi) &= \frac{A - C}{B} \\ &= \frac{5 - 7}{-2\sqrt{3}} \\ &= \frac{-2}{-2\sqrt{3}} \\ &= \frac{1}{\sqrt{3}} \quad \text{or} \quad \frac{\sqrt{3}}{3}\end{aligned}$$

$$\begin{aligned}&\Leftrightarrow \tan(2\phi) = \sqrt{3} \\ &\Leftrightarrow 2\phi = 60^\circ \quad (\text{Given that } 0 < 2\phi < 180^\circ.) \\ &\Leftrightarrow \phi = 30^\circ\end{aligned}$$

The angle of rotation is $\mathbf{30}^\circ$.

Find $\sin\phi$ and $\cos\phi$:

$$\sin\phi = \sin 30^\circ = \frac{1}{2}$$

$$\cos\phi = \cos 30^\circ = \frac{\sqrt{3}}{2}$$

Use the rotation of axes formulas:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$x = \frac{\sqrt{3}}{2}x' - \frac{1}{2}y' \quad \Rightarrow \quad x = \frac{1}{2}\left(\sqrt{3}x' - y'\right)$$

$$y = \frac{1}{2}x' + \frac{\sqrt{3}}{2}y' \quad \Rightarrow \quad y = \frac{1}{2}\left(x' + \sqrt{3}y'\right)$$

Substitute into the given equation:

$$5x^2 - 2\sqrt{3}xy + 7y^2 - 24 = 0$$

$$5\left[\frac{1}{2}\left(\sqrt{3}x' - y'\right)\right]^2 - 2\sqrt{3}\left[\frac{1}{2}\left(\sqrt{3}x' - y'\right)\right]\left[\frac{1}{2}\left(x' + \sqrt{3}y'\right)\right] + 7\left[\frac{1}{2}\left(x' + \sqrt{3}y'\right)\right]^2 - 24 = 0$$

For convenience, let's omit the prime notation for now.

$$5 \cdot \frac{1}{4}\left(3x^2 - 2\sqrt{3}xy + y^2\right) - 2\sqrt{3} \cdot \frac{1}{4} \cdot \left(\sqrt{3}x^2 + 3xy - xy - \sqrt{3}y^2\right) + \frac{7}{4}\left(x^2 + 2\sqrt{3}xy + 3y^2\right) - 24 =$$

Let's multiply through by 4.

$$5\left(3x^2 - 2\sqrt{3}xy + y^2\right) - 2\sqrt{3}\left(\sqrt{3}x^2 + 2xy - \sqrt{3}y^2\right) + 7\left(x^2 + 2\sqrt{3}xy + 3y^2\right) - 96 = 0$$

$$15x^2 - 10\sqrt{3}xy + 5y^2 - 6x^2 - 4\sqrt{3}xy + 6y^2 + 7x^2 + 14\sqrt{3}xy + 21y^2 - 96 = 0$$

$$16x^2 + 32y^2 = 96$$

Divide through by 96.

$$\frac{x^2}{6} + \frac{y^2}{3} = 1$$

Put in the prime notation.

$$\mathbf{\frac{(x')^2}{6} + \frac{(y')^2}{3} = 1}$$