MATH 151 POP QUIZ II: SOLUTIONS

REVIEW FOR CHAPTER 10

Find the following limits.

Write ∞ or $-\infty$ when appropriate. If a limit does not exist, and ∞ and $-\infty$ are inappropriate, write "DNE" (Does Not Exist).

1) $\lim\limits_{x\to 3} \dfrac{x^2-9}{x-3} \quad \dfrac{\to 0}{\to 0}$

As x approaches 3, both the numerator and the denominator approach 0. We have the indeterminate form $\frac{0}{0}$ at $x = 3$. (Indeterminate forms require further analysis on our part.) The "Factoring and Canceling" trick works here.

$$\lim_{x\to 3} \frac{x^2-9}{x-3} = \lim_{x\to 3} \frac{(x+3)(x-3)}{x-3}$$

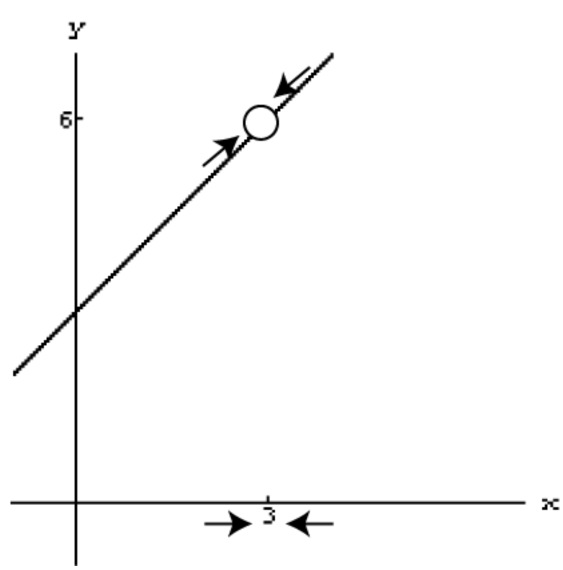
We can cancel the $(x-3)$ factors, because we can assume that $x \neq 3$ if we are considering a limit as x approaches 3.

$$= \lim_{x\to 3} (x+3)$$

We can plug in $x = 3$ now.

$$= 3+3$$
$$= 6$$

Note: The graph of $f(x) = \dfrac{x^2-9}{x-3}$ looks like the graph of $g(x) = x+3$, except that there is a removable discontinuity at $x = 3$ (note that f is undefined there).



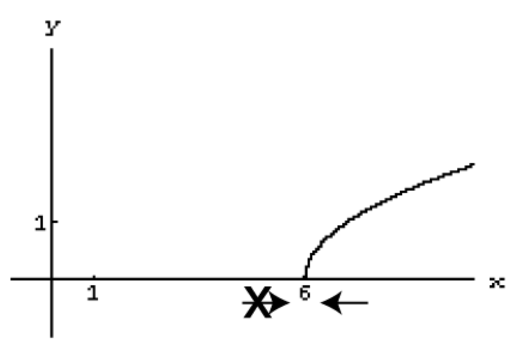
2) $\lim\limits_{x \to 0} \dfrac{\sqrt{9-x}-3}{x} \quad \dfrac{\to 0}{\to 0}$

Again, we have the indeterminate form $\frac{0}{0}$. We will do some preliminary manipulation of the given expression; you sometimes have to do this when trig expressions are involved. This time, we will "Rationalize the Numerator."

$$\begin{aligned}
\lim_{x \to 0} \frac{\sqrt{9-x}-3}{x} &= \lim_{x \to 0} \frac{\left(\sqrt{9-x}-3\right)}{(x)} \bullet \frac{\left(\sqrt{9-x}+3\right)}{\left(\sqrt{9-x}+3\right)} \\
&= \lim_{x \to 0} \frac{\left(\sqrt{9-x}\right)^2 - (3)^2}{x\left(\sqrt{9-x}+3\right)} \\
&= \lim_{x \to 0} \frac{(9-x)-9}{x\left(\sqrt{9-x}+3\right)} \\
&= \lim_{x \to 0} \frac{\overbrace{-x}^{-1}}{\underbrace{x}_{1}\left(\sqrt{9-x}+3\right)} \\
&= \lim_{x \to 0} \frac{-1}{\sqrt{9-x}+3} \\
&= \frac{-1}{\sqrt{9-(0)}+3} \\
&= -\frac{1}{6}
\end{aligned}$$

3) $\lim\limits_{x \to 6} \sqrt{x-6}$

DNE, because the radicand $x-6$ stays negative as x approaches 6 from the left (i.e., through lower numbers; consider 5.9, 5.99, 5.999, …), and $\sqrt{x-6}$ does not evaluate as a real number. It is true that the right-hand limit is 0: $\lim\limits_{x \to 6^+} \sqrt{x-6} = 0$, but the left-hand limit $\lim\limits_{x \to 6^-} \sqrt{x-6}$ DNE. Notice that plugging in $x = 6$ actually doesn't work here.



4) $\lim_{x\to\infty} \dfrac{2x^3 - x + 3}{4x^2 + 1} \quad \dfrac{\to\infty}{\to\infty}$

Here, we have the indeterminate form $\dfrac{\infty}{\infty}$.

We have a rational expression written as a quotient of polynomials, and we are investigating the "long-term behavior" of this function (i.e., the limit as x approaches ∞ or $-\infty$). Let's divide the numerator and the denominator by the highest power of x that appears in the denominator.

$$\lim_{x\to\infty} \frac{2x^3 - x + 3}{4x^2 + 1} = \lim_{x\to\infty} \frac{\dfrac{2x^3}{x^2} - \dfrac{x}{x^2} + \dfrac{3}{x^2}}{\dfrac{4x^2}{x^2} + \dfrac{1}{x^2}}$$

$$= \lim_{x\to\infty} \frac{\overbrace{2x}^{\to\infty} - \overbrace{\dfrac{1}{x}}^{\to 0} + \overbrace{\dfrac{3}{x^2}}^{\to 0}}{4 + \underbrace{\dfrac{1}{x^2}}_{\to 0}} \quad \frac{\to\infty}{\to 4}$$

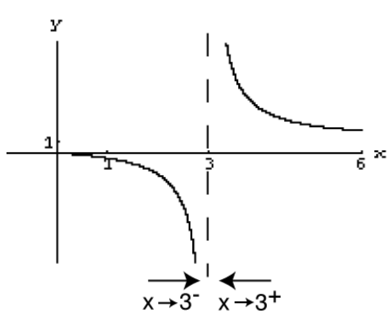
$$= \infty$$

5) $\lim_{x\to 3^+} \dfrac{\overbrace{x}^{\to 3}}{\underbrace{x - 3}_{\substack{\to 0^+ \\ \text{(stays} \\ \text{positive)}}}} = \infty$

6) $\lim_{x\to 3^-} \dfrac{\overbrace{x}^{\to 3}}{\underbrace{x - 3}_{\substack{\to 0^- \\ \text{(stays} \\ \text{negative)}}}} = -\infty$

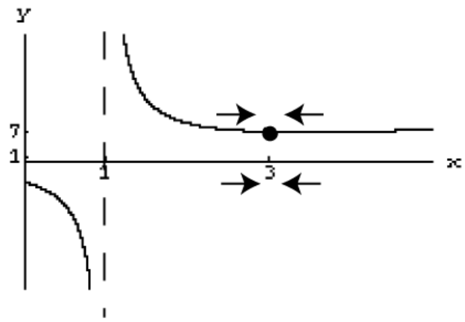
7) $\lim_{x\to 3} \dfrac{x}{x - 3}$ DNE, because the left-hand and right-hand limits do not match.

5-7 Note) Here is the graph of $f(x) = \dfrac{x}{x - 3}$:



8) $\lim_{x \to 3} \frac{x^2+5}{x-1} = \frac{(3)^2+5}{(3)-1} = \frac{14}{2} = 7$

Plugging in $x = 3$ works, because the function $f(x) = \frac{x^2+5}{x-1}$ is continuous ("unbroken") at $x = 3$. (Limits are used to define continuity, but we get the idea….)



9) $\lim_{x \to -3^-} \frac{1}{x^2+3x}$

$$\lim_{x \to -3^-} \frac{1}{x^2+3x} = \lim_{x \to -3^-} \frac{1}{\underbrace{x}_{\to -3}\underbrace{(x+3)}_{\to 0^-}} \quad \frac{\to 1}{\to 0^+}$$

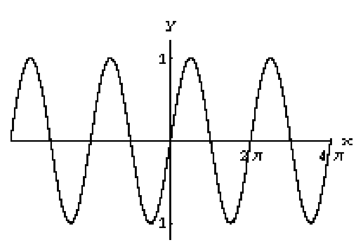
$$= \infty$$

10) $\lim_{x \to 3} \frac{\overbrace{x+4}^{\to 7}}{\underbrace{(x-3)^2}_{0^+}} = \infty$ (The denominator is a square; it stays positive for all $x \neq 3$.)

Contrast #10 with #7.

11) $\lim_{x \to \frac{\pi}{6}} \sin x = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$ ($\sin x$ is continuous at $\frac{\pi}{6}$, so plugging in $\frac{\pi}{6}$ works.)

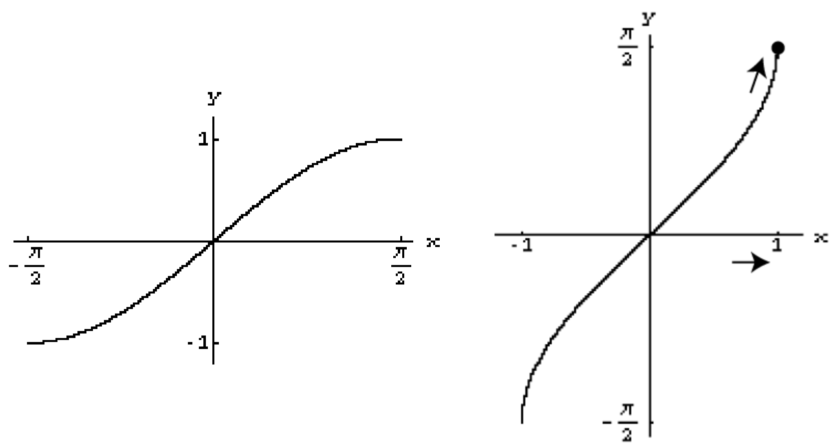
12) $\lim_{x \to \infty} \sin x$ DNE, because $\sin x$ does not approach a single real number as $x \to \infty$.



$f(x) = \sin x$

13) $\lim_{x \to 1^-} \sin^{-1} x = \frac{\pi}{2}$

It may help to graph the arcsin (or "inverse sine") function.
Left figure: We start with the piece of the sin function from $x = -\frac{\pi}{2}$ to $x = \frac{\pi}{2}$.
Right figure: Switch *x*- and *y*-coordinates to get the graph of the arcsin function.



14) $\lim_{x \to 0} \frac{\sin x}{x} = 1$

The proof for this is on pp.119-120 in the textbook (don't worry about the proof).
This statement is used to help show that $D_x(\sin x) = \cos x$ and $D_x(\cos x) = -\sin x$.

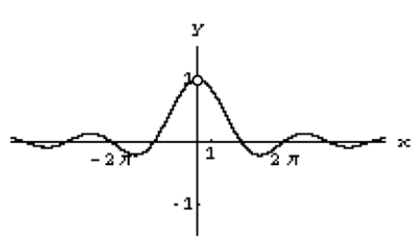
15) $\lim_{x \to \infty} \frac{\sin x}{x} = 0$

We can use a modified version (for the case $x \to \infty$) of the Sandwich or "Squeeze" Theorem on p.64. Observe that $-1 \le \sin x \le 1$ for all real *x*. For all $x > 0$,

$$\underbrace{\frac{-1}{x}}_{\to 0} \le \underbrace{\frac{\sin x}{x}}_{\substack{\text{So,} \\ \to 0}} \le \underbrace{\frac{1}{x}}_{\to 0}$$

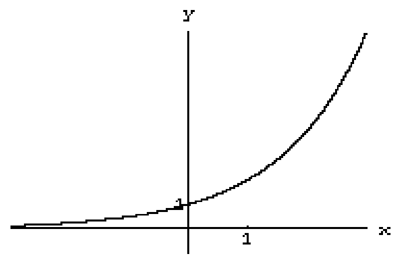
(sin *x* is bounded between –1 and 1, but *x* explodes.)

14-15 Note) Here is the graph of $f(x) = \frac{\sin x}{x}$:



16) $\lim_{x\to\infty} 2^x = \infty$ (2^x is a function that represents exponential growth.)

Graph of $f(x) = 2^x$:

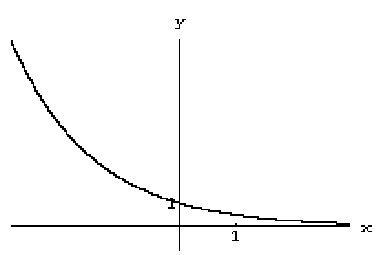


17) $\lim_{x\to\infty} 2^{-x} = \lim_{x\to\infty} \frac{1}{\underbrace{2^x}_{\to\infty}} = 0$

18) $\lim_{x\to\infty} \left(\frac{1}{2}\right)^x = \lim_{x\to\infty} \left(2^{-1}\right)^x = \lim_{x\to\infty} 2^{-x} = 0$, or $\lim_{x\to\infty} \left(\frac{1}{2}\right)^x = \lim_{x\to\infty} \left(\frac{1}{2^x}\right) = 0$ (See #17.)

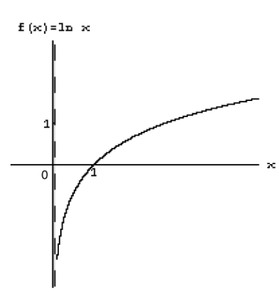
$\left(\frac{1}{2}\right)^x$ is a function that represents exponential decay.

Graph of $f(x) = \left(\frac{1}{2}\right)^x$:



19) $\lim_{x\to\infty} \frac{1}{\frac{1}{x}} = \lim_{x\to\infty} x = \infty$

20) $\lim_{x\to\infty} \ln x = \infty$



21) $\lim_{x\to 0^+} \ln x = -\infty$