QUIZ 1 (CHAPTER 9) SOLUTIONS

MATH 151 – SPRING 2004 – KUNIYUKI
105 POINTS TOTAL, BUT 100 POINTS = 100%

For 1) through 8), evaluate the integrals.

1) $\int e^{2x}\sin(3x)\,dx$ (14 points)

METHOD 1

Alternating signs	Successive derivatives	Successive antiderivatives
$+$	$\sin(3x)$	e^{2x} (strike)
$-$	$3\cos(3x)$	$\frac{1}{2}e^{2x}$
$+$	$-9\sin(3x)$	$\frac{1}{4}e^{2x}$

$\int$

$$\underbrace{\int e^{2x}\sin(3x)\,dx}_{\text{Call "}I\text{"}} = \frac{1}{2}e^{2x}\sin(3x) - \frac{3}{4}e^{2x}\cos(3x) - \frac{9}{4}\underbrace{\int e^{2x}\sin(3x)\,dx}_{I}$$

$$I = \frac{1}{2}e^{2x}\sin(3x) - \frac{3}{4}e^{2x}\cos(3x) - \frac{9}{4}I$$

$$\frac{13}{4}I = \frac{1}{2}e^{2x}\sin(3x) - \frac{3}{4}e^{2x}\cos(3x) + C$$

$$I = \frac{4}{13}\left[\frac{1}{2}e^{2x}\sin(3x) - \frac{3}{4}e^{2x}\cos(3x)\right] + C$$

$$I = \boxed{\frac{2}{13}e^{2x}\sin(3x) - \frac{3}{13}e^{2x}\cos(3x) + C}$$

METHOD 2

Alternating signs	Successive derivatives	Successive antiderivatives
$+$	e^{2x}	$\sin(3x)$ (strike)
$-$	$2e^{2x}$	$\left[-\frac{1}{3}\cos(3x)\right]$
$+$	$4e^{2x}$	$\left[-\frac{1}{9}\sin(3x)\right]$
	$\int$	

$$\underbrace{\int e^{2x}\sin(3x)\,dx}_{\text{Call "}I\text{"}} = -\frac{1}{3}e^{2x}\cos(3x) + \frac{2}{9}e^{2x}\sin(3x) - \frac{4}{9}\underbrace{\int e^{2x}\sin(3x)\,dx}_{I}$$

$$I = -\frac{1}{3}e^{2x}\cos(3x) + \frac{2}{9}e^{2x}\sin(3x) - \frac{4}{9}I$$

$$\frac{13}{9}I = -\frac{1}{3}e^{2x}\cos(3x) + \frac{2}{9}e^{2x}\sin(3x) + C$$

$$I = \frac{9}{13}\left[-\frac{1}{3}e^{2x}\cos(3x) + \frac{2}{9}e^{2x}\sin(3x)\right] + C$$

$$I = \boxed{-\frac{\mathbf{3}}{\mathbf{13}}e^{2x}\cos(3x) + \frac{\mathbf{2}}{\mathbf{13}}e^{2x}\sin(3x) + C}$$

(Note that this is equivalent to our previous result.)

2) $\int \cot^6 x \csc^4 x\, dx$ (10 points)

We like the fact that the csc power is even.
We will let u be the "other guy," namely $\cot x$.

$$u = \cot x$$

$$du = -\underbrace{\csc^2 x}_{\text{Peel}}\,dx$$

$$= \int \cot^6 x \csc^2 x \cdot \csc^2 x\, dx$$

We want to use a Pythagorean identity to transform $\csc^2 x$ and make more "u"s.

$$= \int \cot^6 x\left(1+\cot^2 x\right)\cdot\csc^2 x\, dx$$

Remember, $u = \cot x$

$$du = -\csc^2 x\, dx$$

$$= -\int u^6\left(1+u^2\right)du$$

$$= -\int \left(u^6 + u^8\right) du$$

$$= -\left[\frac{u^7}{7} + \frac{u^9}{9}\right] + C$$

$$= -\frac{u^7}{7} - \frac{u^9}{9} + C$$

Go back to x.

$$= \boxed{-\frac{1}{7}\cot^7 x - \frac{1}{9}\cot^9 x + C}$$

3) $\int \sin^2 x \, dx$ (6 points)

Use a Power - Reducing Identity (PRI).

$$= \int \frac{1 - \cos(2x)}{2} dx$$

$$= \frac{1}{2} \int \left[1 - \cos(2x)\right] dx$$

$$= \frac{1}{2}\left[x - \frac{1}{2}\sin(2x)\right] + C$$

$$= \boxed{\frac{1}{2}x - \frac{1}{4}\sin(2x) + C}$$

4) $\int \sin^{-1} x \, dx$ (8 points)

Use integration by parts. The tabular method is fine, but it does not improve much on the classic method:

Warning: $\sin^{-1} x \neq \dfrac{1}{\sin x}$.

Let $u = \sin^{-1} x$ Let $dv = dx$

$du = \dfrac{1}{\sqrt{1 - x^2}} dx$ $v = x$

$$\int \sin^{-1} x \, dx = uv - \int v \, du$$

$$= x \sin^{-1} x - \int \frac{x}{\sqrt{1 - x^2}} dx$$

Let $u = 1 - x^2$

$du = -2x\, dx$

$-\frac{1}{2}\, du = x\, dx$

$$\int \frac{x}{\sqrt{1-x^2}}\, dx = -\frac{1}{2}\int \frac{du}{\sqrt{u}}$$
$$= -\frac{1}{2}\int u^{-1/2}\, du$$
$$= -\frac{1}{2}\left[\frac{u^{1/2}}{1/2}\right] + C_1$$
$$= -\sqrt{1-x^2} + C_1$$

Combine our results: $\int \sin^{-1} x\, dx = \boxed{\boldsymbol{x\sin^{-1}x + \sqrt{1-x^2} + C}}$

5) $\int \frac{4x-3}{x^2 - 10x + 25}\, dx$ (14 points)

Factor the denominator and find the appropriate partial fraction decomposition form.

$$\frac{4x-3}{x^2-10x+25} \quad \frac{\qquad}{\leftarrow \text{Factor}}$$
$$\frac{4x-3}{(x-5)^2} = \frac{A}{x-5} + \frac{B}{(x-5)^2}$$

Find A and B:

Multiply both sides by $(x-5)^2$.

$$4x - 3 = A(x-5) + B \qquad (\leftarrow \text{Basic equation})$$
$$4x - 3 = Ax - 5A + B$$
$$4x - 3 = \underbrace{(A)}_{=4}x + \underbrace{(-5A + B)}_{=-3}$$

Match corresponding coefficients of like terms.

$$\begin{cases} A = 4 \\ -5A + B = -3 \end{cases}$$

We know $A = 4$. Substitute into the second equation to find B.

$$-5A + B = -3$$
$$-5(4) + B = -3$$
$$-20 + B = -3$$
$$B = 17$$

Partial Fraction Decomposition:

$$\frac{4x-3}{(x-5)^2} = \frac{4}{x-5} + \frac{17}{(x-5)^2}$$

$$\int \frac{4x-3}{x^2-10x+25}\,dx = \int \frac{4}{x-5}\,dx + \int \frac{17}{(x-5)^2}\,dx$$

Let $u = x-5$

$du = dx$

$$= 4\int \frac{du}{u} + 17\int \frac{du}{u^2}$$

$$= 4\ln|u| + 17\int u^{-2}\,du$$

$$= 4\ln|x-5| + 17\left(\frac{u^{-1}}{-1}\right) + K$$

$$= 4\ln|x-5| - \frac{17}{u} + K$$

$$= \boxed{\mathbf{4\ln|x-5| - \frac{17}{x-5} + K}}$$

6) $\displaystyle\int \frac{\sqrt{9+x^2}}{x^4}\,dx$ (20 points)

Use a trig substitution.

$x = 3\tan\theta$ $\left(\text{Actually, the substitution is } \theta = \tan^{-1}\left(\frac{x}{3}\right).\right)$

$dx = 3\sec^2\theta\,d\theta$

$$= \int \frac{\sqrt{9+(3\tan\theta)^2}}{(3\tan\theta)^4}\cdot 3\sec^2\theta\ d\theta$$

Simplify the numerator:

$$\begin{aligned}\sqrt{9+(3\tan\theta)^2} &= \sqrt{9+9\tan^2\theta}\\ &= \sqrt{9(1+\tan^2\theta)}\\ &= 3\sqrt{1+\tan^2\theta}\\ &= 3\sqrt{\sec^2\theta}\\ &= 3|\sec\theta|\\ &= 3\sec\theta\end{aligned}$$

We can remove the absolute value symbols, because it is assumed (based on our trig sub) that θ is in Quadrant I or IV.

$$= \int \frac{3\sec\theta}{81\tan^4\theta} \cdot 3\sec^2\theta \, d\theta$$

$$= \int \frac{9\sec^3\theta}{81\tan^4\theta} \, d\theta$$

$$= \frac{1}{9} \int \frac{\frac{1}{\cos^3\theta}}{\frac{\sin^4\theta}{\cos^4\theta}} \, d\theta$$

$$= \frac{1}{9} \int \frac{1}{\cos^3\theta} \cdot \frac{\cos^4\theta}{\sin^4\theta} \, d\theta$$

$$= \frac{1}{9} \int \frac{\cos\theta}{\sin^4\theta} \, d\theta$$

$$= \frac{1}{9} \int \frac{du}{u^4}$$

$$= \frac{1}{9} \int u^{-4} \, du$$

$$= \frac{1}{9} \cdot \frac{u^{-3}}{-3} + C$$

$$= -\frac{1}{27u^3} + C$$

$$= -\frac{1}{27\sin^3\theta} + C$$

$$= -\frac{1}{27}\csc^3\theta + C$$

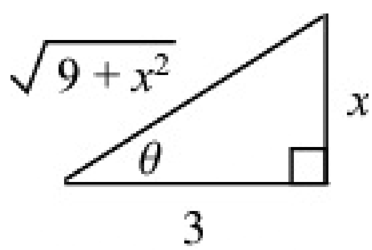
Perform a standard u-sub:

Let $u = \sin\theta$

$du = \cos\theta \, d\theta$

Find $\csc\theta$:

$$x = 3\tan\theta \Rightarrow \tan\theta = \frac{x}{3}$$



Notice that our missing side length is in the original integrand.

$$\sin\theta = \frac{x}{\sqrt{9+x^2}}$$

$$\csc\theta = \frac{\sqrt{9+x^2}}{x}$$

Answer = $\boxed{-\frac{1}{27}\left(\frac{\sqrt{9+x^2}}{x}\right)^3 + C \quad \text{or} \quad -\frac{\left(9+x^2\right)^{3/2}}{27x^3} + C}$

7) $\int \frac{\ln x}{x(\ln x - 5)}\,dx$ (8 points)

Perform a standard u-sub:

Let $u = \ln x - 5 \quad \Rightarrow \quad \ln x = u + 5$

$du = \frac{1}{x}\,dx$

$$= \int \frac{u+5}{u}\,du$$

$$= \int \left(1 + \frac{5}{u}\right) du$$

$$= u + 5\ln|u| + C$$

$$= (\ln x - 5) + 5\ln|\ln x - 5| + C \quad \text{(The "}+C\text{" can absorb the "-5.")}$$

$$= \boxed{\mathbf{\ln x + 5\ln|\ln x - 5| + C}}$$

8) $\int \frac{x}{\sqrt{16x - x^2}}\,dx$ (20 points)

Complete the square in the radicand:

$$\sqrt{16x - x^2} = \sqrt{-(x^2 - 16x)}$$

$$= \sqrt{-\left(x^2 - 16x \underbrace{+64}_{\text{You're actually throwing in -64.}}\right) + 64}$$

$$= \sqrt{64 - (x^2 - 16x + 64)}$$

$$= \sqrt{64 - (x-8)^2}$$

Trig sub: We have the form $\sqrt{a^2 - u^2}$, which leads to the sub $u = a\sin\theta$.

$$x - 8 = 8\sin\theta$$

$$x = 8\sin\theta + 8$$

$$dx = 8\cos\theta\,d\theta$$

$$= \int \frac{x}{\sqrt{64-(x-8)^2}}\, dx$$

$$= \int \frac{8\sin\theta + 8}{\sqrt{64-(8\sin\theta)^2}} \cdot 8\cos\theta \, d\theta$$

Simplify the denominator:

$$\begin{aligned}
\sqrt{64-(8\sin\theta)^2} &= \sqrt{64-64\sin^2\theta} \\
&= \sqrt{64(1-\sin^2\theta)} \\
&= 8\sqrt{1-\sin^2\theta} \\
&= 8\sqrt{\cos^2\theta} \\
&= 8|\cos\theta| \\
&= 8\cos\theta
\end{aligned}$$

We can remove the absolute value symbols, because it is assumed (based on our trig sub) that θ is in Quadrant I or IV.

$$= \int \frac{8\sin\theta + 8}{8\cos\theta} \cdot 8\cos\theta \, d\theta$$

$$= \int (8\sin\theta + 8) \, d\theta$$

$$= -8\cos\theta + 8\theta + C$$

Find θ:

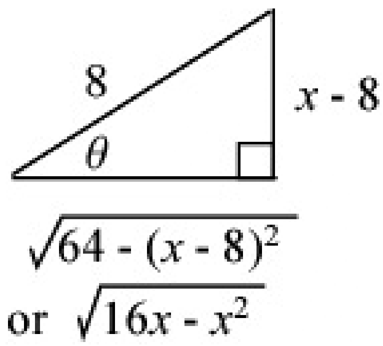
$$x - 8 = 8\sin\theta$$

$$\frac{x-8}{8} = \sin\theta$$

$$\theta = \sin^{-1}\left(\frac{x-8}{8}\right)$$

Find $\cos\theta$:

$$\sin\theta = \frac{x-8}{8}$$



Notice that our missing side length is in the original integrand.

$$\cos\theta = \frac{\sqrt{16x-x^2}}{8}$$

Combine our results:

$$= -8\left(\frac{\sqrt{16x-x^2}}{8}\right) + 8\sin^{-1}\left(\frac{x-8}{8}\right) + C$$

$$= \boxed{-\sqrt{16x-x^2} + 8\sin^{-1}\left(\frac{x-8}{8}\right) + C}$$

9) We want to integrate $\int \frac{1}{(x+9)^2(x^2+9)^2}\,dx$ using partial fractions.

Write the <u>form</u> of the partial fraction decomposition for the integrand, $\frac{1}{(x+9)^2(x^2+9)^2}$. You do <u>not</u> have to work out the integral. (5 points)

$$\boxed{\frac{A}{x+9} + \frac{B}{(x+9)^2} + \frac{Cx+D}{x^2+9} + \frac{Ex+F}{(x^2+9)^2}}$$