# QUIZ 2 (CHAPTER 10) SOLUTIONS

**MATH 151 – SPRING 2004 – KUNIYUKI**
**105 POINTS TOTAL, BUT 100 POINTS = 100%**

1) Find the limits. Write $\infty$ or $-\infty$ when appropriate. If a limit does not exist, and $\infty$ and $-\infty$ are inappropriate, write "DNE" (Does Not Exist). Indicate indeterminate forms whenever appropriate, though you don't have to indicate signs for them. Indicate when you are applying L'Hôpital's Rule.

a) $\lim\limits_{x\to -6} \dfrac{\sqrt{x+55}-7}{x^2-36}$

$$\lim_{x\to -6} \frac{\sqrt{x+55}-7}{x^2-36} \qquad \frac{\to \sqrt{-6+55}-7=\sqrt{49}-7=0}{\to 36-36=0} \quad \left(\frac{0}{0}\right)$$

$$\overset{L'H}{=} \lim_{x\to -6} \frac{D_x\left[(x+55)^{1/2}-7\right]}{D_x\left(x^2-36\right)}$$

$$= \lim_{x\to -6} \frac{\frac{1}{2}(x+55)^{-1/2}(1)}{2x}$$

$$= \lim_{x\to -6} \frac{1}{4x\sqrt{x+55}}$$

$$= \frac{1}{4(-6)\sqrt{-6+55}}$$

$$= \frac{1}{-24\sqrt{49}}$$

$$= \frac{1}{-24(7)}$$

$$= \boxed{-\frac{1}{168}}$$

b) $\lim_{x\to 0} \frac{x+\tan x}{\sin x}$

$$\lim_{x\to 0} \frac{x+\tan x}{\sin x} \qquad \frac{\to 0+\tan 0=0}{\to \sin 0=0} \quad \left(\frac{0}{0}\right)$$

$$\overset{L'H}{=} \lim_{x\to 0} \frac{1+\sec^2 x}{\cos x}$$

$$= \frac{1+(\sec 0)^2}{\cos 0}$$

$$= \frac{1+(1)^2}{1}$$

$$= \boxed{\mathbf{2}}$$

c) $\lim_{x\to 0^+} \frac{x^2+2}{\sin x}$

$$\lim_{x\to 0^+} \frac{x^2+2}{\sin x} \qquad \frac{\to 2}{\to 0^+}$$

$$= \boxed{\infty}$$

Do <u>not</u> use L'Hôpital's Rule!

d) $\lim_{x\to 0^+} (\csc x - \cot x)$

$$\lim_{x\to 0^+} (\csc x - \cot x)$$

$$= \lim_{x\to 0^+} \left(\frac{1}{\sin x} - \frac{\cos x}{\sin x}\right) \qquad (\infty - \infty)$$

$$= \lim_{x\to 0^+} \frac{1-\cos x}{\sin x} \qquad \frac{\to 1-\cos 0 = 1-1=0}{\to \sin 0 = 0} \quad \left(\frac{0}{0}\right)$$

$$\overset{L'H}{=} \lim_{x\to 0^+} \frac{\sin x}{\cos x}$$

$$= \lim_{x\to 0^+} \tan x$$

$$= \tan 0$$

$$= \boxed{\mathbf{0}}$$

e) $\lim_{x \to \infty} x^{1/x}$

$$\lim_{x \to \infty} x^{1/x} \qquad \left(\infty^0\right)$$

Let $y = x^{1/x}$

$$\ln y = \ln x^{1/x}$$

$$\ln y = \frac{1}{x} \ln x \qquad \left(x > 0 \text{ "eventually" as } x \to \infty.\right)$$

$$\lim_{x \to \infty} \ln y = \lim_{x \to \infty} \underbrace{\frac{1}{x}}_{\to 0} \underbrace{\ln x}_{\to \infty} \qquad (0 \cdot \infty)$$

$$= \lim_{x \to \infty} \frac{\ln x}{x} \qquad \left(\frac{\infty}{\infty}\right)$$

$$\overset{L'H}{=} \lim_{x \to \infty} \frac{1/x}{1}$$

$$= \lim_{x \to \infty} \frac{1}{x}$$

$$= 0$$

Then,

$$\lim_{x \to \infty} x^{1/x} = \lim_{x \to \infty} y$$

$$= \lim_{x \to \infty} e^{\ln y}$$

$$= e^0$$

$$= \boxed{\mathbf{1}}$$

2) Indicate whether the integral converges or diverges. If it converges, find its value. Either way, show all work and use good form, as in class!

a) $\int_{-\infty}^{-3} e^{3x+2}\, dx$ (8 points)

$$\int_{-\infty}^{-3} \underbrace{e^{3x+2}}_{\substack{\text{continuous} \\ \text{on } (-\infty,-3]}} dx = \lim_{t\to-\infty} \int_{t}^{-3} e^{3x+2}\, dx \qquad (\text{if the limit exists})$$

$$= \lim_{t\to-\infty} \left[\frac{1}{3}e^{3x+2}\right]_t^{-3}$$

$$= \lim_{t\to-\infty} \left( \left[\frac{1}{3}e^{3(-3)+2}\right] - \underbrace{\left[\frac{1}{3}e^{\overbrace{3t+2}^{\to-\infty}}\right]}_{\to 0} \right)$$

$$= \frac{1}{3}e^{-7} - 0$$

$$= \boxed{\frac{1}{3}e^{-7}}$$

Does the above integral **converge** or diverge?

b) $\int_{-\infty}^{\infty} \frac{3x}{\left(x^2+1\right)^4}\, dx$

We need to make a cut - at 0, say.

$$\int_{-\infty}^{\infty} \underbrace{\frac{3x}{\left(x^2+1\right)^4}}_{\substack{\text{everywhere} \\ \text{continuous}}} dx = \underbrace{\int_{-\infty}^{0} \frac{3x}{\left(x^2+1\right)^4}\, dx}_{\text{I}} + \underbrace{\int_{0}^{\infty} \frac{3x}{\left(x^2+1\right)^4}\, dx}_{\text{II}} \qquad (\text{if the integrals converge})$$

Indefinite integral:

$$\int \frac{3x}{\left(x^2+1\right)^4}\, dx$$

Let $u = x^2 + 1$

$$du = 2x\, dx \quad \Rightarrow \quad x\, dx = \frac{1}{2}\, du$$

$$\int \frac{3x}{\left(x^2+1\right)^4}\,dx = \int \frac{3\cdot\frac{1}{2}\,du}{u^4}$$

$$= \frac{3}{2}\int u^{-4}\,du$$

$$= \frac{3}{2}\left[\frac{u^{-3}}{-3}\right] + C$$

$$= -\frac{1}{2u^3} + C$$

$$= -\frac{1}{2\left(x^2+1\right)^3} + C$$

<u>Work out integral II</u>:

$$\int_0^\infty \frac{3x}{\left(x^2+1\right)^4}\,dx = \lim_{t\to\infty}\int_0^t \frac{3x}{\left(x^2+1\right)^4}\,dx \quad \text{(if the limit exists)}$$

$$= \lim_{t\to\infty}\left[-\frac{1}{2\left(x^2+1\right)^3}\right]_0^t$$

$$= \lim_{t\to\infty}\left(\left[-\frac{1}{2\left(t^2+1\right)^3}\right] - \left[-\frac{1}{2\left((0)^2+1\right)^3}\right]\right)$$

$$= \lim_{t\to\infty}\left(\underbrace{\left[-\frac{1}{\underbrace{2\left(t^2+1\right)^3}_{\to\infty}}\right]}_{\to 0} - \left[-\frac{1}{2}\right]\right)$$

$$= 0 + \frac{1}{2}$$

$$= \frac{1}{2}$$

Work out integral I:

$$\int_{-\infty}^{0} \frac{3x}{\left(x^2+1\right)^4}\,dx = \lim_{t\to-\infty} \int_{t}^{0} \frac{3x}{\left(x^2+1\right)^4}\,dx \quad \left(\text{if the limit exists}\right)$$

$$= \lim_{t\to-\infty} \left[ -\frac{1}{2\left(x^2+1\right)^3} \right]_t^0$$

$$= \lim_{t\to-\infty} \left( \left[ -\frac{1}{2\left((0)^2+1\right)^3} \right] - \left[ -\frac{1}{2\left(t^2+1\right)^3} \right] \right)$$

$$= \lim_{t\to-\infty} \left( \left[ -\frac{1}{2} \right] - \underbrace{\left[ -\frac{1}{2\underbrace{\left(t^2+1\right)^3}_{\to\infty}} \right]}_{\to 0} \right)$$

$$= -\frac{1}{2}$$

Evaluate the original integral:

$$\int_{-\infty}^{\infty} \frac{3x}{\left(x^2+1\right)^4}\,dx = I + II$$

$$= -\frac{1}{2} + \frac{1}{2}$$

$$= \boxed{\mathbf{0}}$$

Does the above integral **converge** or diverge?

c) $\int_5^\infty \frac{1}{\sqrt{x-5}}\,dx$

We need to make a cut - at 30, say. (Note that 30 – 5 is a perfect square, which makes for a nice radicand.)

$$\underbrace{\int_5^\infty \frac{1}{\sqrt{x-5}}}_{\substack{\text{continuous}\\ \text{on }(5,\infty)}}\,dx = \underbrace{\int_5^{30} \frac{1}{\sqrt{x-5}}\,dx}_{\text{I}} + \underbrace{\int_{30}^\infty \frac{1}{\sqrt{x-5}}\,dx}_{\text{II}} \quad \left(\text{if the integrals converge}\right)$$

Indefinite integral:

$$\int \frac{1}{\sqrt{x-5}}\,dx$$

$$\text{Let } u = x-5$$
$$du = dx$$

$$\begin{aligned}\int \frac{1}{\sqrt{x-5}}\,dx &= \int \frac{du}{\sqrt{u}} \\ &= \int u^{-1/2}\,du \\ &= \left[\frac{u^{1/2}}{1/2}\right] + C \\ &= 2u^{1/2} + C \\ &= 2\sqrt{x-5} + C\end{aligned}$$

Work out integral II:

$$\begin{aligned}\int_{30}^\infty \frac{1}{\sqrt{x-5}}\,dx &= \lim_{t\to\infty} \int_{30}^t \frac{1}{\sqrt{x-5}}\,dx \quad \left(\text{if the limit exists}\right) \\ &= \lim_{t\to\infty} \left[2\sqrt{x-5}\right]_{30}^t \\ &= \lim_{t\to\infty}\left(\underbrace{\left[2\sqrt{t-5}\right]}_{\to\infty} - \underbrace{\left[2\sqrt{30-5}\right]}_{\substack{\to 2\sqrt{25}\\ =10}}\right) \\ &= \infty\end{aligned}$$

Does the above integral converge or **diverge**?

Note: Integral I, in fact, converges, but that is irrelevant at this point!

d) $\int_0^{33} (x-1)^{-1/5}\, dx$

We need to make a cut at 1 because of the discontinuity there.
Note that the integrand is continuous everywhere else.

$$\int_0^{33} (x-1)^{-1/5}\, dx = \underbrace{\int_0^1 (x-1)^{-1/5}\, dx}_{\text{I}} + \underbrace{\int_1^{33} (x-1)^{-1/5}\, dx}_{\text{II}} \quad \text{(if the integrals converge)}$$

Indefinite integral:

$$\int (x-1)^{-1/5}\, dx$$

Let $u = x-1$
$du = dx$

$$\begin{aligned}\int (x-1)^{-1/5}\, dx &= \int u^{-1/5}\, du \\ &= \left[\frac{u^{4/5}}{4/5}\right] + C \\ &= \frac{5}{4}u^{4/5} + C \\ &= \frac{5}{4}(x-1)^{4/5} + C\end{aligned}$$

Work out integral II:

$$\begin{aligned}\int_1^{33} (x-1)^{-1/5}\, dx &= \lim_{t\to 1^+} \int_t^{33} (x-1)^{-1/5}\, dx \quad \text{(if the limit exists)} \\ &= \lim_{t\to 1^+} \left[\frac{5}{4}(x-1)^{4/5}\right]_t^{33} \\ &= \lim_{t\to 1^+} \left( \underbrace{\left[\frac{5}{4}(33-1)^{4/5}\right]}_{\substack{\to \frac{5}{4}\left(\sqrt[5]{32}\right)^4 \\ = \frac{5}{4}(2)^4 \\ =20}} - \underbrace{\left[\frac{5}{4}(t-1)^{4/5}\right]}_{\substack{\to \frac{5}{4}(0)^{4/5} \\ =0}} \right) \\ &= 20\end{aligned}$$

Work out integral I:

$$\int_0^1 (x-1)^{-1/5}\,dx = \lim_{t\to 1^-} \int_0^t (x-1)^{-1/5}\,dx \quad (\text{if the limit exists})$$

$$= \lim_{t\to 1^-} \left[ \frac{5}{4}(x-1)^{4/5} \right]_0^t$$

$$= \lim_{t\to 1^-} \left( \underbrace{\left[ \frac{5}{4}(t-1)^{4/5} \right]}_{\to \frac{5}{4}(0)^{4/5} = 0} - \underbrace{\left[ \frac{5}{4}(0-1)^{4/5} \right]}_{\to \frac{5}{4}(-1)^{4/5} = \frac{5}{4}(1) = \frac{5}{4}} \right)$$

$$= -\frac{5}{4}$$

Evaluate the original integral:

$$\int_0^{33} (x-1)^{-1/5}\,dx = I + II$$

$$= -\frac{5}{4} + 20$$

$$= \boxed{\mathbf{\frac{75}{4}} \text{ or } \mathbf{18.75}}$$

Does the above integral **converge** or diverge?