QUIZ 3 (SECTIONS 11.1-11.5) SOLUTIONS

MATH 151 – SPRING 2004 – KUNIYUKI
105 POINTS TOTAL, BUT 100 POINTS = 100%

1) Find the limits. Write ∞ or $-\infty$ when appropriate. If a limit does not exist, and ∞ and $-\infty$ are inappropriate, write "DNE" (Does Not Exist). You do not have to show work. (8 points total; 4 points each)

a) $\lim_{n\to\infty} a_n$, where $a_n = 4 + (-1)^n$

DNE, because the terms of the sequence 3, 5, 3, 5, 3, 5, … are not approaching a single real number.

b) $\lim_{n\to\infty} a_n$, where $a_n = 7 - \left(\frac{8}{9}\right)^n$

7, because $\left(\frac{8}{9}\right)^n \to 0$ as $n \to \infty$. (Think: Geometric sequence with $|r| = \left|\frac{8}{9}\right| = \frac{8}{9} < 1$.)

2) Does the series $\sum_{n=1}^{\infty} \left(\frac{3}{\sqrt{n}} - \frac{4}{n^2} \right)$ converge or diverge? Circle one:
(You do not have to show work.)

Converges **Diverges**

(4 points)

The series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ diverges, because it is a p-series with $p = \frac{1}{2} \leq 1$.

The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, because it is a p-series with $p = 2 > 1$.

The given series is a linear combination of the above two series (with nonzero weights 3 and –4), so the given series diverges.

3) Find the sum of the series $\sum_{n=1}^{\infty}\left(4^n 5^{3-n}\right)$. (12 points)

$$\sum_{n=1}^{\infty}\left(4^n 5^{3-n}\right)=\sum_{n=1}^{\infty}\left(4^n \cdot 5^3 \cdot 5^{-n}\right)$$
$$=\sum_{n=1}^{\infty}\frac{4^n \cdot 125}{5^n}$$
$$=\sum_{n=1}^{\infty}125\left(\frac{4}{5}\right)^n$$
$$=\sum_{n=1}^{\infty}125\left(\frac{4}{5}\right)\left(\frac{4}{5}\right)^{n-1}$$
$$=\sum_{n=1}^{\infty}100\left(\frac{4}{5}\right)^{n-1}$$

The sum, S, of the geometric series $\sum_{n=1}^{\infty} ar^{n-1}$ $\left(|r|<1\right)$ is given by $S=\frac{a}{1-r}$.

$$S=\frac{a}{1-r}$$
$$=\frac{100}{1-\frac{4}{5}}$$
$$=\frac{100}{1/5}$$
$$=\mathbf{500}$$

4) State the Alternating Series Test for the series $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$, where all $a_n>0$. (6 points)

[The given series is clearly an alternating series.]

If the $\{a_n\}$ sequence is nonincreasing, and if $a_n \to 0$ as $n \to \infty$, then the given series converges.

i.e., If $0<a_{k+1}\le a_k$ for all $k \ge 1$, and if $\lim_{n\to\infty} a_n = 0$, then the given series converges.

5) The series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{3}{n^{2/3}}$ is … (circle one:)
(You do not have to show work.)

Absolutely Convergent **Conditionally Convergent** Divergent

(4 points)

The series can be shown to converge by the Alternating Series Test (AST). It is an alternating series of the form $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$, where all $a_n > 0$. We know that $\frac{3}{n^{2/3}}$ is nonincreasing for $n \geq 1$, and that $\frac{3}{n^{2/3}} \to 0$.

However, the corresponding "absolute value series" $\sum_{n=1}^{\infty}\frac{3}{n^{2/3}}$ diverges, because it is a nonzero constant multiple of the series $\sum_{n=1}^{\infty}\frac{1}{n^{2/3}}$, which diverges by the *p*-series test $\left(p = \frac{2}{3} \leq 1\right)$.

6) True or False: If $\sum_{n=1}^{\infty}a_n$ is a convergent positive-term series, then $\sum_{n=1}^{\infty}(-1)^n a_n$ must also be a convergent series. Circle one:
(You do not have to show work.)

True False

(4 points)

This is true by the Absolute Convergence Test (ACT). The series $\sum_{n=1}^{\infty}(-1)^n a_n$ must converge if its corresponding "absolute value series" $\sum_{n=1}^{\infty}a_n$ converges.

Problems 7) and 8) are based on our discussions in class. Assume that all hypotheses for the tests are satisfied, and you apply the tests correctly.
You do <u>not</u> have to show work. (8 points; 4 points each)

7) You use the Basic Comparison Test, and the test shows that the series $\sum_{n=1}^{\infty} a_n$ converges. When you try to use the Ratio Test on this series, you find that $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$. Is this possible? Circle one:

Yes, it is possible. No, it is not.

The Ratio Test would be "inconclusive" in this case, which would not be inconsistent with the result of convergence from the BCT. In fact, this often happens with algebraic a_n.

8) You use the Integral Test, and the test shows that the series $\sum_{n=1}^{\infty} b_n$ diverges. When you try to use the Root Test on this series, you find that $\lim_{n\to\infty}\sqrt[n]{|b_n|} = 0.8$. Is this possible? Circle one:

Yes, it is possible. **No, it is not.**

The Root Test would imply convergence in this case. This contradicts the result from the Integral Test.

9) For each of the following series:

- Determine whether it converges (write "C") or diverges (write "D").
- Whenever you use a test for convergence/divergence, name it. (You may abbreviate as in class.)
- Whenever you use the Integral Test, state the assumptions (hypotheses) for the test and verify as we have done in class. Set up and work out the integral using good form.
- Show work (as suggested in class). You may ask me if you can write a particular statement without proof.
- You may use the back of the test if you run out of room. Write "SEE BACK" and indicate the problem you're working on.

(59 points total)

a) $\sum_{n=3}^{\infty} \frac{n^{4/3}}{n^2 - 2}$ (8 points)

We have a positive-term series.

$\frac{n^{4/3}}{n^2 - 2}$ is an algebraic expression. The Ratio and Root Tests fail; they yield $L = 1$.

Method 1: Basic Comparison Test (BCT)

Compare $\frac{n^{4/3}}{n^2 - 2}$ with $\frac{n^{4/3}}{n^2} = \frac{n^{4/3}}{n^{6/3}} = \frac{1}{n^{2/3}}$.

(Consider dominant terms in the numerator and the denominator.)

$$\frac{n^{4/3}}{n^2 - 2} \geq \frac{n^{4/3}}{n^2} \quad \left(= \frac{1}{n^{2/3}} \right) \qquad \text{(for all } n \geq 3\text{)}$$

The inequality holds, because raising the denominator shrinks the overall value of an expression, given that the numerators are the same, and all numerators and denominators are positive for $n \geq 3$.

We know that the "little brother" series $\sum_{n=3}^{\infty} \frac{1}{n^{2/3}}$ diverges, since it is a p-series with $p = \frac{2}{3} \leq 1$. Then, the "big brother" series $\sum_{n=3}^{\infty} \frac{n^{4/3}}{n^2 - 2}$ must also **diverge (D)**.

Method 2: Limit Comparison Test (LCT)

Once again, use $\sum_{n=3}^{\infty} \frac{1}{n^{2/3}}$ as the comparison series.

$$\lim_{n \to \infty} \frac{\frac{n^{4/3}}{n^2 - 2}}{\frac{1}{n^{2/3}}} = \lim_{n \to \infty} \left[\frac{n^{4/3}}{n^2 - 2} \cdot n^{2/3} \right]$$

$$= \lim_{n \to \infty} \frac{n^2}{n^2 - 2} \qquad \left(\text{form } \frac{\text{poly}}{\text{poly}}; \text{ same degree in num., denom.} \right)$$

$$= 1 \qquad \left(\text{ratio of leading coefficients in num., denom.} \right)$$

1 is a positive real number, and we know $\sum_{n=3}^{\infty} \frac{1}{n^{2/3}}$ diverges (truncated p-series; $p = \frac{2}{3} \leq 1$), so $\sum_{n=3}^{\infty} \frac{n^{4/3}}{n^2 - 2}$ must also **diverge (D)**.

b) $\sum_{n=1}^{\infty} \frac{8^n}{(2n)!}$ (12 points)

The factorial hints at the Ratio Test.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{8^{n+1}}{[2(n+1)]!}}{\frac{8^n}{(2n)!}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{8^{n+1}}{(2n+2)!}\cdot\frac{(2n)!}{8^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{8^{n+1}}{8^n}\cdot\frac{(2n)!}{(2n+2)!}\right|$$

$$= \lim_{n\to\infty}\left|8\cdot\underbrace{\frac{1}{(2n+2)(2n+1)}}_{\to 0}\right|$$

$$= 0$$

$$< 1$$

Therefore, the given series **converges (C)**.

c) $\sum_{n=1}^{\infty} \frac{5^n}{n+4^n}$ (Use a Comparison Test.) (12 points)

We have a positive-term series.

Note: The Basic Comparison Test (BCT) would have failed (if you had used the strategy suggested in class), because $\frac{5^n}{n+4^n} \le \frac{5^n}{4^n} = \left(\frac{5}{4}\right)^n$ for all $n \ge 1$. When applying the BCT, you can't use a divergent "big brother" series to verify that the "little brother" series diverges.

Let's use the Limit Comparison Test (LCT).

Compare $\frac{5^n}{n+4^n}$ with $\frac{5^n}{4^n} = \left(\frac{5}{4}\right)^n$.

(Consider dominant terms in the numerator and the denominator, as we did when we attempted the BCT.)

$$\lim_{n\to\infty} \frac{\frac{5^n}{n+4^n}}{\frac{5^n}{4^n}} = \lim_{n\to\infty} \left(\frac{5^n}{n+4^n} \cdot \frac{4^n}{5^n} \right)$$

$$= \lim_{n\to\infty} \frac{4^n}{n+4^n}$$

$$= \lim_{n\to\infty} \frac{\frac{4^n}{4^n}}{\frac{n}{4^n} + \frac{4^n}{4^n}}$$

$$= \lim_{n\to\infty} \frac{1}{\underbrace{\frac{n}{4^n}}_{\to 0} + 1}$$

$$= 1$$

1 is a positive real number, and we know $\sum_{n=1}^{\infty} \left(\frac{5}{4} \right)^n$ diverges (geometric series; $|r| = \left| \frac{5}{4} \right| = \frac{5}{4} \geq 1$), so $\sum_{n=1}^{\infty} \frac{5^n}{n+4^n}$ must also **diverge (D)**.

d) $\sum_{n=1}^{\infty} \cos(\pi n)$ (4 points)

Notice that $\cos(\pi n) = (-1)^n$ (for any integer *n*), which has no limit as $n \to \infty$. By the <u>n^{th}-Term Test for Divergence</u>, the series **diverges (D)**.

e) $\sum_{n=4}^{\infty} \frac{1}{n(\ln n)^{3/2}}$ (Use the Integral Test.) (23 points)

State the assumptions (hypotheses) for the test and verify as we have done in class. Set up and work out the integral using good form.

Let $f(x) = \frac{1}{x(\ln x)^{3/2}}$; this is our continuous interpolating function for $x \geq 4$.

Check the hypotheses of the test:

- f is positive-valued on $[4, \infty)$.

  Note that $\ln x > \ln 1 = 0$ for $x \geq 4$.

- f is continuous on $[4, \infty)$.

- f decreases on $[4, \infty)$. Verify:

$$f(x) = \frac{1}{x(\ln x)^{3/2}}$$

By the Reciprocal Rule (which is obtained from the Quotient Rule),

$$f'(x) = -\frac{D_x\left[x(\ln x)^{3/2}\right]}{\left[x(\ln x)^{3/2}\right]^2} \qquad \leftarrow \underline{\text{Use the Product Rule.}}$$

$$= -\frac{\overbrace{\left[D_x(x)\right]}^{=1} \cdot \left[(\ln x)^{3/2}\right] + [x] \cdot D_x\left[(\ln x)^{3/2}\right]}{\left[x(\ln x)^{3/2}\right]^2}$$

$$= -\frac{(\ln x)^{3/2} + [x] \cdot \left[\frac{3}{2}(\ln x)^{1/2} \cdot \frac{1}{x}\right]}{x^2(\ln x)^3}$$

$$= -\frac{(\ln x)^{3/2} + \frac{3}{2}(\ln x)^{1/2}}{x^2(\ln x)^3}$$

Again, note that $\ln x > \ln 1 = 0$ for $x \geq 4$.

$$f'(x) < 0 \quad \text{on} \quad [4, \infty)$$

Indefinite integral:

$$\int \frac{1}{x(\ln x)^{3/2}}\, dx$$

Let $u = \ln x$

$$du = \frac{1}{x}\, dx$$

$$= \int \frac{1}{u^{3/2}}\, du$$
$$= \int u^{-3/2}\, du$$
$$= \frac{u^{-1/2}}{-1/2} + C$$
$$= -\frac{2}{\sqrt{u}} + C$$
$$= -\frac{2}{\sqrt{\ln x}} + C$$

Now:

$$\int_4^\infty \frac{1}{x(\ln x)^{3/2}}\, dx = \lim_{t\to\infty} \int_4^t \frac{1}{x(\ln x)^{3/2}}\, dx$$
$$= \lim_{t\to\infty} \left[-\frac{2}{\sqrt{\ln x}} \right]_4^t$$
$$= \lim_{t\to\infty} \left(\underbrace{\left[-\frac{2}{\underbrace{\sqrt{\ln t}}_{\to\infty}} \right]}_{\to 0} - \left[-\frac{2}{\sqrt{\ln 4}} \right] \right)$$
$$= \frac{2}{\sqrt{\ln 4}} \quad \text{(a real number)}$$

This integral converges, so the given series **converges (C)**.