QUIZ 4 (SECTIONS 11.6-11.8) SOLUTIONS

MATH 151 – SPRING 2004 – KUNIYUKI

PART 1: GRADED OUT OF 80 POINTS; SCORE CUT IN HALF (80 → 40)
PART 2: 65 POINTS
TOTAL ON PARTS 1 AND 2: 105 POINTS, BUT 100 POINTS = 100%

(PART 1)

No notes, books, or calculators!

Fill in the table below. You may use the back for [ungraded] scratch work. Simplify where appropriate, but you do not have to compute factorials.

$f(x)$	First four nonzero terms of the Maclaurin series	Summation notation form for the Maclaurin series	Interval of convergence, I, for the Maclaurin series
$\sin x$	$x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots$	$\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$	$(-\infty,\infty)$
$\cos x$	$1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots$	$\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}$	$(-\infty,\infty)$
$\sinh x$	$x+\frac{x^3}{3!}+\frac{x^5}{5!}+\frac{x^7}{7!}+\ldots$	$\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}$	$(-\infty,\infty)$
$\cosh x$	$1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\ldots$	$\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}$	$(-\infty,\infty)$
$\tan^{-1} x$	$x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\ldots$	$\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$	$[-1,1]$
e^x	$1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots$	$\sum_{n=0}^{\infty}\frac{x^n}{n!}$	$(-\infty,\infty)$
$\ln(1+x)$	$x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots$	$\sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{n+1}$, or $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$	$(-1,1]$
$\frac{1}{1-x}$	$1+x+x^2+x^3+\ldots$	$\sum_{n=0}^{\infty}x^n$	$(-1,1)$

(PART 2)

Show all work, simplify as appropriate, and use "good form and procedure" (as in class).
Box in your final answers!
No notes or books allowed. A scientific calculator is allowed.

1) Consider the series $\sum_{n=1}^{\infty} \frac{5n}{3^n}(x+2)^n$. (27 points total)

a) What is the center of this series? $\boxed{\mathbf{-2.}}$ It is c in the form $\sum a_n(x-c)^n$.

b) Find the interval of convergence. Show all work, as in class!

Let $u_n = \frac{5n}{3^n}(x+2)^n$.

$$L = \lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{\frac{5(n+1)}{3^{n+1}}(x+2)^{n+1}}{\frac{5n}{3^n}(x+2)^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{5(n+1)}{3^{n+1}}\cdot\frac{3^n}{5n}(x+2)\right|$$

$$= \lim_{n\to\infty}\left|\underbrace{\frac{5n+5}{5n}}_{\to 1}\cdot\underbrace{\frac{3^n}{3^{n+1}}}_{=\frac{1}{3}}(x+2)\right|$$

$$= \frac{1}{3}|x+2|$$

We know the series converges when $L<1$ (and diverges when $L>1$).

$$\frac{1}{3}|x+2|<1$$

$$|x+2|<3$$

Solve the absolute value inequality:

$$\begin{array}{rcccl} -3 & < & x+2 & < & 3 \\ -3-2 & < & x & < & 3-2 \\ -5 & < & x & < & 1 \end{array}$$

We know that the series converges for these values of x.

Check $x=-5$:

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{5n}{3^n}(x+2)^n &= \sum_{n=1}^{\infty} \frac{5n}{3^n}(-5+2)^n \\ &= \sum_{n=1}^{\infty} \frac{5n}{3^n}(-3)^n \\ &= \sum_{n=1}^{\infty} \frac{5n}{3^n}(-1)^n \cdot 3^n \\ &= \sum_{n=1}^{\infty} (-1)^n 5n \end{aligned}$$

This series diverges by the nth-Term Test for Divergence, because as $n \to \infty$, $5n \to \infty$ and $(-1)^n 5n$ has a DNE limit.

Check $x=1$:

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{5n}{3^n}(x+2)^n &= \sum_{n=1}^{\infty} \frac{5n}{3^n}(1+2)^n \\ &= \sum_{n=1}^{\infty} \frac{5n}{3^n}(3)^n \\ &= \sum_{n=1}^{\infty} 5n \end{aligned}$$

This series diverges by the nth-Term Test for Divergence, because as $n \to \infty$, $5n \to \infty$.

Answer: $\boxed{\boldsymbol{I} = (-5, 1)}$

2) Use summation notation to answer the following. (10 points total)

a) Find a power series representation for $f(x)=\dfrac{1}{2+9x}$, $|x|<\dfrac{2}{9}$.

Use the Geometric Template.

$$\begin{aligned}\frac{1}{2+9x}&=\frac{1}{2}\cdot\frac{1}{1+\frac{9}{2}x}\\&=\frac{1}{2}\cdot\frac{1}{1-\left(-\frac{9}{2}x\right)}\\&=\frac{1}{2}\cdot\sum_{n=0}^{\infty}\left(-\frac{9}{2}x\right)^n \qquad ,\left|-\frac{9}{2}x\right|<1\\&=\frac{1}{2}\cdot\sum_{n=0}^{\infty}(-1)^n\left(\frac{9}{2}\right)^n x^n \qquad ,\frac{9}{2}|x|<1\\&=\boxed{\sum_{n=0}^{\infty}(-1)^n\frac{9^n}{2^{n+1}}x^n \qquad ,|x|<\frac{2}{9}}\end{aligned}$$

b) Use part a) to find a power series representation for $D_x\left(\dfrac{1}{2+9x}\right)$, $|x|<\dfrac{2}{9}$.

For $|x|<\dfrac{2}{9}$,

$$\begin{aligned}D_x\left(\frac{1}{2+9x}\right)&=D_x\left[\sum_{n=0}^{\infty}(-1)^n\frac{9^n}{2^{n+1}}x^n\right]\\&=\sum_{n=0}^{\infty}D_x\left[(-1)^n\frac{9^n}{2^{n+1}}x^n\right]\\&=\sum_{n=0}^{\infty}(-1)^n\frac{9^n}{2^{n+1}}\cdot D_x\left[x^n\right]\\&=\boxed{\sum_{n=1}^{\infty}(-1)^n\frac{9^n}{2^{n+1}}\cdot nx^{n-1} \quad \text{or} \quad \sum_{n=1}^{\infty}(-1)^n\frac{n9^n}{2^{n+1}}x^{n-1}}\end{aligned}$$

<u>Note</u>: The $n = 0$ term in the series you're differentiating is a constant, so its derivative is 0.

3) Evaluate $\int x^3 \arctan x^5 \, dx, \left| x \right| < 1$. Hint: The Maclaurin series for arctan x is $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$. Just use series; don't use integration by parts. (12 points)

$$\arctan x^5 = \sum_{n=0}^{\infty} (-1)^n \frac{\left(x^5\right)^{2n+1}}{2n+1}$$
$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{10n+5}}{2n+1}$$

$$x^3 \arctan x^5 = \sum_{n=0}^{\infty} (-1)^n \frac{x^3 x^{10n+5}}{2n+1}$$
$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{10n+8}}{2n+1}$$

$$\int x^3 \arctan x^5 \, dx = \int \sum_{n=0}^{\infty} (-1)^n \frac{x^{10n+8}}{2n+1} \, dx$$
$$= \sum_{n=0}^{\infty} \int (-1)^n \frac{x^{10n+8}}{2n+1} \, dx$$
$$= \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{x^{10n+9}}{(2n+1)(10n+9)} + C}$$

4) Find the first four terms of the Taylor series for $f(x) = 4^x$ at $c = 2$. (Assume that such a series exists.) (16 points)

$$f(x) = 4^x \qquad f(2) = 4^2 = 16$$
$$f'(x) = 4^x \ln 4 \qquad f'(2) = 4^2 \ln 4 = 16 \ln 4$$
$$f''(x) = 4^x (\ln 4)^2 \qquad f''(2) = 4^2 (\ln 4)^2 = 16 (\ln 4)^2$$
$$f'''(x) = 4^x (\ln 4)^3 \qquad f'''(2) = 4^2 (\ln 4)^3 = 16 (\ln 4)^3$$

Taylor series at $c = 2$:

$$f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f'''(2)}{3!}(x-2)^3 + \ldots$$
$$= 16 + (16 \ln 4)(x-2) + \frac{16(\ln 4)^2}{2}(x-2)^2 + \frac{16(\ln 4)^3}{6}(x-2)^3 + \ldots$$
$$= \boxed{16 + (16 \ln 4)(x-2) + 8(\ln 4)^2 (x-2)^2 + \frac{8}{3}(\ln 4)^3 (x-2)^3 + \ldots}$$