QUIZ 5 (CHAPTER 13) SOLUTIONS

MATH 151 – SPRING 2004 – KUNIYUKI
105 POINTS TOTAL, BUT 100 POINTS = 100%

Show all work, simplify as appropriate, and use "good form and procedure" (as in class).
Box in your final answers!
No notes, books, or calculators allowed.

When graphing, be reasonably accurate, and clearly indicate orientation. Use as many arrowheads as appropriate. Clearly indicate *x*- and *y*-intercepts, endpoints, and extreme points when feasible.

1) A plane curve *C* is described by:

$$x = 8\cos t + 2$$
$$y = 6\sin t - 4$$
$$0 \le t < 2\pi$$

Find a corresponding rectangular equation in *x* and *y*. (7 points)

$$x = 8\cos t + 2 \quad \Rightarrow \quad \cos t = \frac{x-2}{8}$$
$$y = 6\sin t - 4 \quad \Rightarrow \quad \sin t = \frac{y+4}{6}$$

Use a Pythagorean identity:

$$\cos^2 t + \sin^2 t = 1$$

$$\boxed{\left(\frac{x-2}{8}\right)^2 + \left(\frac{y+4}{6}\right)^2 = 1}$$

<u>Note</u>: *C* is an ellipse centered at (2, -4) oriented counterclockwise.

2) A plane curve C is described by:

$$x = -2\sin t$$
$$y = 4 + 2\sin t$$
$$0 \le t \le \pi$$

Sketch the graph of C using the grid below. Use arrowheads and labels to clearly indicate orientation for any relevant value of t. (10 points)

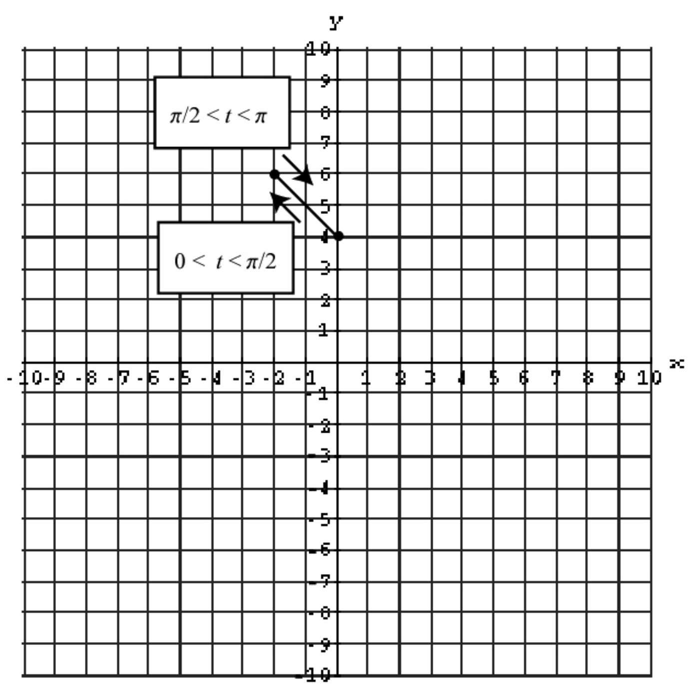
Eliminate the parameter:

$$y = 4 + 2\sin t$$
$$y = 4 - (-2\sin t)$$
$$y = 4 - x$$
$$x + y = 4 \quad \text{(line)}$$

What part of the line do we pick up for C? What about orientation?

As t increases from 0 to $\frac{\pi}{2}$, x decreases from 0 down to –2, and y increases from 4 to 6. The "particle" starts at (0,4) at $t = 0$ and goes up and to the left until it reaches (-2,6).

As t increases from $\frac{\pi}{2}$ to π, x increases from –2 up to 0, and y decreases from 6 to 4. From the point (-2,6) at $t = \frac{\pi}{2}$, the particle goes down and to the right until it reaches (0,4) again.



3) (For parts a) through e).) A plane curve C is described by:

$$x = e^{4t}$$
$$y = 2t^3 - 1$$
$$-5 \le t \le 10$$

(38 points total)

a) Find the slope of the tangent line at the point on the curve that corresponds to $t = 2$.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dy/dt}{dx/dt} \\
&= \frac{6t^2}{4e^{4t}} \\
&= \frac{3t^2}{2e^{4t}} \\
\left[\frac{dy}{dx}\right]_{t=2} &= \frac{3(2)^2}{2e^{4(2)}} \\
&= \frac{12}{2e^8} \\
&= \boxed{\frac{\mathbf{6}}{\boldsymbol{e}^{\mathbf{8}}}}
\end{aligned}$$

b) Find the point(s) on C at which the tangent line is horizontal. If there are none, write "NONE." Box in your final answer(s).

Find when $\frac{dy}{dt} = 0$:

$$6t^2 = 0$$
$$t = 0, \text{ which is in } [-5, 10]$$

The corresponding point is:

$$\begin{aligned}
(x, y) &= \left(e^{4(0)}, 2(0)^3 - 1\right) \\
&= \left(e^0, 0 - 1\right) \\
&= \boxed{(\mathbf{1}, -\mathbf{1})}
\end{aligned}$$

Note: $\frac{dx}{dt} \ne 0$ here.

c) Find the point(s) on C at which the tangent line is vertical. If there are none, write "NONE." Box in your final answer(s).

Find when $\frac{dx}{dt} = 0$:

$4e^{4t}$ is never 0, so the answer is: **NONE**.

d) Set up, **but do not evaluate**, an integral that represents the length of C. Don't leave any general notation that can be replaced by something more specific.

$$\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \boxed{\int_{-5}^{10} \sqrt{\left(4e^{4t}\right)^2 + \left(6t^2\right)^2}\, dt}$$

e) Find $\frac{d^2y}{dx^2}$ in terms of t. Simplify completely.

From part a),

$$\frac{dy}{dx} = \frac{3t^2}{2e^{4t}} \quad (= y')$$

Now,

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}$$

$$= \frac{D_t\left(\frac{3t^2}{2e^{4t}}\right)}{4e^{4t}}$$

Use the Quotient Rule to get the numerator.

$$= \frac{\frac{\left(2e^{4t}\right)\cdot D_t\left(3t^2\right) - \left(3t^2\right)\cdot D_t\left(2e^{4t}\right)}{\left(2e^{4t}\right)^2}}{4e^{4t}}$$

$$= \frac{\frac{\left(2e^{4t}\right)\cdot\left(6t\right) - \left(3t^2\right)\cdot\left(8e^{4t}\right)}{4e^{8t}}}{4e^{4t}}$$

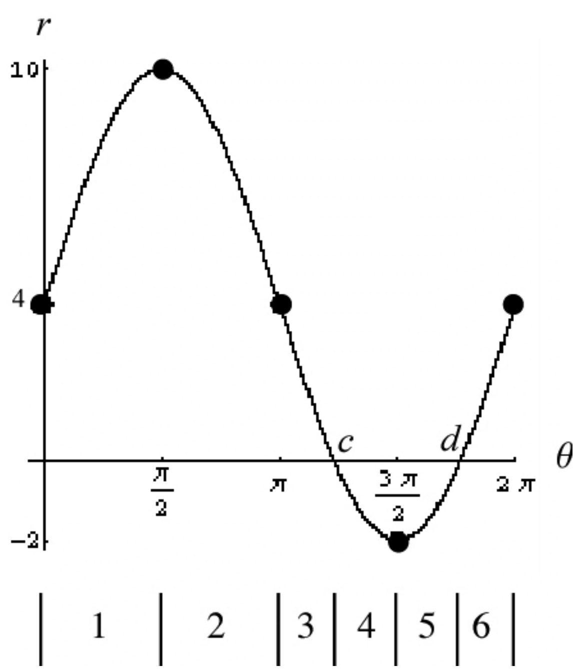
$$= \frac{12te^{4t} - 24t^2e^{4t}}{16e^{12t}}$$

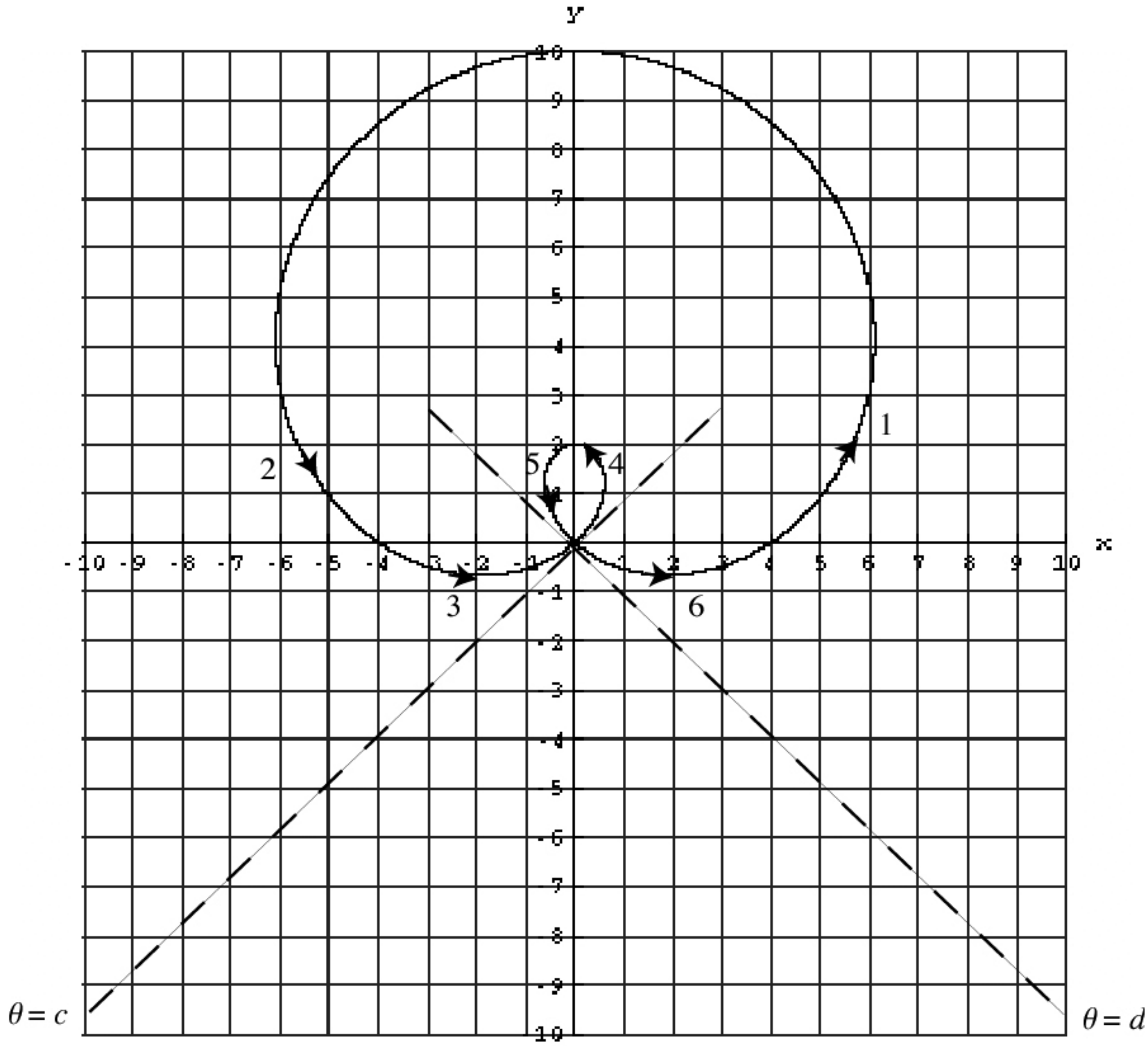
$$= \frac{12te^{4t}(1-2t)}{16e^{12t}}$$

$$= \boxed{\frac{3t(1-2t)}{4e^{8t}}}$$

4) Sketch the graph of $r = 4 + 6\sin\theta$ using the grid below. You <u>must</u> first carefully graph r against θ as Cartesian/rectangular coordinates; identify key values of r and θ, as in class. You do <u>not</u> have to determine the exact value(s) of θ for which $r = 0$. (20 points)

Graph r against θ as Cartesian coordinates:





5) Find the slope of the tangent line to the graph of the polar equation $r = 4 + 6\sin\theta$ (same as in Problem #4) at the point corresponding to $\theta = \dfrac{\pi}{6}$. Give an exact answer. (18 points)

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

$$= \frac{\dfrac{d}{d\theta}(r\sin\theta)}{\dfrac{d}{d\theta}(r\cos\theta)} \quad \frac{\leftarrow \text{Use Product Rule}}{\leftarrow \text{Use Product Rule}}$$

$$= \frac{\dfrac{dr}{d\theta}\sin\theta + r\cos\theta}{\dfrac{dr}{d\theta}\cos\theta - r\sin\theta}$$

$$r = 4 + 6\sin\theta$$

$$\frac{dr}{d\theta} = 6\cos\theta$$

$$\left[\frac{dr}{d\theta}\right]_{\theta=\frac{\pi}{6}} = 6\cos\frac{\pi}{6}$$

$$= 6\left(\frac{\sqrt{3}}{2}\right)$$

$$= 3\sqrt{3}$$

We will also plug in:

$$\left(\theta = \frac{\pi}{6}\right) \Rightarrow \quad \sin\theta = \sin\frac{\pi}{6} = \frac{1}{2}$$

$$\cos\theta = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

$$r = 4 + 6\sin\theta = 4 + 6\sin\frac{\pi}{6} = 4 + 6\left(\frac{1}{2}\right) = 4 + 3 = 7$$

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

$$= \frac{\left(3\sqrt{3}\right)\left(\frac{1}{2}\right) + 7\left(\frac{\sqrt{3}}{2}\right)}{\left(3\sqrt{3}\right)\left(\frac{\sqrt{3}}{2}\right) - 7\left(\frac{1}{2}\right)}$$

$$= \frac{\frac{3\sqrt{3}}{2} + \frac{7\sqrt{3}}{2}}{\frac{9}{2} - \frac{7}{2}}$$

$$= \frac{3\sqrt{3} + 7\sqrt{3}}{9 - 7}$$

$$= \frac{10\sqrt{3}}{2}$$

$$= \boxed{\mathbf{5\sqrt{3}}}$$

6) Find the area of the region $R = \left\{ (r, \theta) : 0 \le \theta \le \frac{\pi}{6},\ 0 \le r \le 3\cos\theta \right\}$. (12 points)

Note: $3\cos\theta \ge 0$ on $\left[0, \frac{\pi}{6}\right]$.

$$A = \int_a^b \frac{1}{2} r^2 \, d\theta$$

$$= \int_0^{\pi/6} \frac{1}{2} (3\cos\theta)^2 \, d\theta$$

$$= \int_0^{\pi/6} \frac{1}{2} \cdot 9\cos^2\theta \, d\theta$$

$$= \frac{9}{2} \int_0^{\pi/6} \cos^2\theta \, d\theta$$

$$= \frac{9}{2} \int_0^{\pi/6} \frac{1 + \cos(2\theta)}{2} \, d\theta$$

$$= \frac{9}{4} \int_0^{\pi/6} [1 + \cos(2\theta)] \, d\theta$$

$$= \frac{9}{4} \left[\theta + \frac{1}{2} \sin(2\theta) \right]_0^{\pi/6}$$

$$= \frac{9}{4} \left(\left[\frac{\pi}{6} + \frac{1}{2} \sin\left(\frac{\pi}{3}\right) \right] - \left[0 + \frac{1}{2} \sin 0 \right] \right)$$

$$= \frac{9}{4} \left(\left[\frac{\pi}{6} + \frac{1}{2} \left(\frac{\sqrt{3}}{2} \right) \right] - [0] \right)$$

$$= \frac{9}{4} \left(\frac{\pi}{6} + \frac{\sqrt{3}}{4} \right)$$

$$= \boxed{\mathbf{\frac{3\pi}{8} + \frac{9\sqrt{3}}{16}}}$$