MATH 245: QUIZ 2 SOLUTIONS

1) a) **{2, 3, 5, 7, 11, 13, 17, 19}**
b) $|T| = \mathbf{8}$

2) a) **False**; $\varnothing$ is not an element of S.
b) **True**; $\varnothing$ is always a subset of any [reasonable] set.
c) If $n = |S| = 4$, then $|P(S)| = 2^n = 2^4 = \mathbf{16}$
d) **True**; $\varnothing$ is always a member of the power set of a [reasonable] set.
e) **True**; $P(S)$ is, itself, a set, so $\varnothing$ is a subset.

3) The cardinality of $A \times B$ is mn, so the cardinality of $P(A \times B)$ is $\mathbf{2^{mn}}$.
One interpretation: Think of a two-dimensional coordinate system in which the elements of A are the possible horizontal coordinates and the elements of B are the possible vertical coordinates. Then, $A \times B$ is the set of all mn possible points in this coordinate system. $P(A \times B)$ then consists of all possible graphs, or point-plots; its cardinality is 2^{mn}, which corresponds to the fact that each of the 2^{mn} points can be either in a given point-plot or not.

4) a) **{1, 3, 5}**, which has the elements that are in A_1 but not in A_2.
b) **{1, 3, 5, 7, 9}**, which has the elements in U that are not in A_2.
c) $A_3 \cup \varnothing = A_3 =$ **{1, 2, 10}**.
d) **{1, 2, 3, 4, 5, 6, 8, 10}**, which is the entire collection of elements from the A_i sets.
e) **{2}**. 2 is the only element that appears in all the A_i sets.

5) (Steps may vary.)

a)

$$\begin{aligned}\overline{A \cap (B \cup \overline{A})} &= \overline{(A \cap B) \cup (A \cap \overline{A})} \\ &= \overline{(A \cap B) \cup \varnothing} \\ &= \overline{(A \cap B)} \\ &= \overline{A} \cup \overline{B} \quad (\text{DeMorgan})\end{aligned}$$

b)

$$\begin{aligned}\overline{(A \cup A) \cap \overline{A}} &= \overline{A \cap \overline{A}} \\ &= \overline{\varnothing} \\ &= U\end{aligned}$$

6) a) **Yes**. f is one-to-one, since different elements in the domain are mapped to different elements in the codomain.

b) **No**. f is not onto, because d does not have a preimage.

c) **No**. An invertible function must be both one-to-one <u>and</u> onto.

7)

a) The elements 0, 1, 2, 3, and 4 in the domain S get mapped to different elements in the codomain **Z**. However, -1 gets mapped to the same element in **Z** as 1 does, and -2 gets mapped to the same element in **Z** as 2 does; this is because f depends on x only through an even power of x, namely x^6. So, the cardinality of the range of f = the number of images in **Z** = **5**.

b) **No**. f is not one-to-one; for example, +1 and -1 get mapped to the same element in **Z** (namely, 3).

c) **No**. f is not onto, since not every integer is an image. The range (which is finite) is clearly not equal to the codomain **Z** (which is infinite).

d) **No**. f can't have an inverse function.

8)

$$\sum_{j=1}^{3}\sum_{k=0}^{j-1}\left\lfloor\frac{j+k}{2}\right\rfloor = \underbrace{\sum_{k=0}^{0}\left\lfloor\frac{1+k}{2}\right\rfloor}_{j=1} + \underbrace{\sum_{k=0}^{1}\left\lfloor\frac{2+k}{2}\right\rfloor}_{j=2} + \underbrace{\sum_{k=0}^{2}\left\lfloor\frac{3+k}{2}\right\rfloor}_{j=3}$$

$$= \underbrace{\left(\left\lfloor\frac{1+0}{2}\right\rfloor\right)}_{j=1} + \underbrace{\left(\left\lfloor\frac{2+0}{2}\right\rfloor + \left\lfloor\frac{2+1}{2}\right\rfloor\right)}_{j=2} + \underbrace{\left(\left\lfloor\frac{3+0}{2}\right\rfloor + \left\lfloor\frac{3+1}{2}\right\rfloor + \left\lfloor\frac{3+2}{2}\right\rfloor\right)}_{j=3}$$

$$= \left(\left\lfloor\frac{1}{2}\right\rfloor\right) + \left(\lfloor 1\rfloor + \left\lfloor\frac{3}{2}\right\rfloor\right) + \left(\left\lfloor\frac{3}{2}\right\rfloor + \lfloor 2\rfloor + \left\lfloor\frac{5}{2}\right\rfloor\right)$$

$$= (0) + (1+1) + (1+2+2)$$

$$= \mathbf{7}$$

A grid could also help:

j \ k	0	1	2
1	$\left\lfloor\frac{1+0}{2}\right\rfloor = 0$		
2	$\left\lfloor\frac{2+0}{2}\right\rfloor = 1$	$\left\lfloor\frac{2+1}{2}\right\rfloor = 1$	
3	$\left\lfloor\frac{3+0}{2}\right\rfloor = 1$	$\left\lfloor\frac{3+1}{2}\right\rfloor = 2$	$\left\lfloor\frac{3+2}{2}\right\rfloor = 2$