MATH 245: QUIZ 3 SOLUTIONS

1) a) $980 = (2)(2)(5)(7)(7) = \mathbf{2^2 \cdot 5 \cdot 7^2}$

b) $616 = (2)(2)(2)(7)(11) = \mathbf{2^3 \cdot 7 \cdot 11}$

c) The prime factors of interest are 2, 5, 7, and 11.
For the lcm, we take the larger exponent on each prime factor.
$980 = 2^2 \cdot 5^1 \cdot 7^2 \cdot 11^0$
$616 = 2^3 \cdot 5^0 \cdot 7^1 \cdot 11^1$
$\text{lcm} = 2^3 \cdot 5^1 \cdot 7^2 \cdot 11^1 = \mathbf{21{,}560}$

d) For the gcd, we take the smaller exponent on each prime factor.
$980 = 2^2 \cdot 5^1 \cdot 7^2 \cdot 11^0$
$616 = 2^3 \cdot 5^0 \cdot 7^1 \cdot 11^1$
$\text{gcd} = 2^2 \cdot 5^0 \cdot 7^1 \cdot 11^0 = 2^2 \cdot 7 = \mathbf{28}$

2)

Let $a = 4743$ and $b = 867$.

$$ab = \gcd(a,b) \cdot \text{lcm}(a,b)$$

$$(4743)(867) = (51) \cdot \text{lcm}(a,b)$$

$$\text{lcm}(a,b) = \frac{(4743)(867)}{51} = \mathbf{80{,}631}$$

3)

Let n be a composite integer.
Then, $n = rs$ for some integers r and s strictly between 1 and n.
If both r and s are greater than $\sqrt{n}$, then
$rs > \sqrt{n}\sqrt{n} = n$, which contradicts "$n = rs$."
So, one or both of r or s must be less than or equal to $\sqrt{n}$.

4)

Think of 100! as $(1)(2)(3)\cdots(100)$. We want to count the number of "3-factors" in the integers from 1 to 100.

There are $\left\lfloor \frac{100}{3} \right\rfloor = 33$ multiples of 3 between 1 and 100. (3, 6, 9, …, 99)
We can pick up one 3-factor from each of these 33 multiples.

There are $\left\lfloor \frac{100}{9} \right\rfloor = 11$ multiples of 9 between 1 and 100. (9, 18, 27, …, 99)
We can pick up a second 3-factor from each of these 11 multiples.

There are $\left\lfloor \frac{100}{27} \right\rfloor = 3$ multiples of 27 between 1 and 100. (27, 54, 81)
We can pick up a third 3-factor from each of these 3 multiples.

There is $\left\lfloor \frac{100}{81} \right\rfloor = 1$ multiple of 81 between 1 and 100. (Just 81, itself)
We can pick up a fourth 3-factor from the 81.

No higher power of 3 will divide an integer from 1 to 100.
So, there are a total of $n = 33+11+3+1 = \mathbf{48}$ "3-factors" in 100!.

5)

When 1000 is divided by 7, the quotient is $\left\lfloor \frac{1000}{7} \right\rfloor$ = 142.

Let a = 1000, d = 7, and q = 142. We want the remainder, r.

$$\begin{aligned} a &= dq + r \\ 1000 &= (7)(142) + r \\ 1000 &= 994 + r \\ r &= \mathbf{6} \end{aligned}$$

6) **Yes**. 3709 - 37 = 3672, which is divisible by 51.

7)

$a \equiv b \pmod{m}$, so $\exists s \in \mathbf{Z}$ such that $a = b + sm$
$c \equiv d \pmod{m}$, so $\exists t \in \mathbf{Z}$ such that $c = d + tm$
Then, $\exists s, t \in \mathbf{Z}$ such that:

$$\begin{aligned} a + c &= (b + sm) + (d + tm) \\ a + c &= (b + d) + (sm + tm) \\ a + c &= (b + d) + m(s + t) \end{aligned}$$

$s + t \in \mathbf{Z}$, so $a + c \equiv b + d \pmod{m}$.

8) $2^0 + 2^3 + 2^6 = 1 + 8 + 64 = \mathbf{73}$

9)

Bit Position Value	Bit	Remainder r
		182
128	1	54
64	0	54
32	1	22
16	1	6
8	0	6
4	1	2
2	1	0
1	0	0

Binary representation (reading down the bits): **10110110**

10)

First, work out the Euclidean Algorithm:

$$59 = 56 \cdot 1 + 3 \quad (A)$$
$$56 = 3 \cdot 18 + 2 \quad (B)$$
$$3 = 2 \cdot 1 + 1 \quad (C)$$

Solve for the remainder in each equation:

$$1 = 3 - 2 \cdot 1 \quad (C^*)$$
$$2 = 56 - 3 \cdot 18 \quad (B^*)$$
$$3 = 59 - 56 \cdot 1 \quad (A^*)$$

Replace the "2" in (C*) with the expression in (B*):

$$1 = 3 - (56 - 3 \cdot 18) \cdot 1$$
$$1 = 3 - 56 + 3 \cdot 18$$
$$1 = 3 \cdot 19 - 56$$

Replace the "3" with the expression in (A*):

$$1 = (59 - 56 \cdot 1) \cdot 19 - 56$$
$$1 = 59 \cdot 19 - 56 \cdot 19 - 56$$
$$1 = 59 \cdot 19 - 56 \cdot 20$$
$$1 = 59 \cdot 19 + 56 \cdot (-20)$$

So, s = 19 and t = -20 will work.

11)

Let's subtract 4 from both sides of the congruence.

$$11x + 4 \equiv 13 \pmod{15}$$
$$11x \equiv 9 \pmod{15} \quad (*)$$

The hint tells us that -4 is a [multiplicative] inverse of 11 (mod 15):

Since 1 and (15)(3) - (11)(4) are the same number, they must have the same remainder when they are divided by 15:

$$1 \equiv (15)(3) - (11)(4) \pmod{15}$$

A term that is a multiple of 15 acts like 0 in mod 15 arithmetic.

$$1 \equiv -(11)(4) \pmod{15}$$
$$1 \equiv (11)(-4) \pmod{15}$$

Now, multiply both sides of (*) by -4.
We know that -4 is an inverse of 11 (mod 15), so we know that the left side will just be $1x$ (mod 15).

$$(-4)(11)x \equiv (-4)(9) \pmod{15}$$
$$x \equiv -36 \pmod{15}$$

So, -36 is a solution to the given congruence. More generally, the congruence class (mod 15) containing -36 consists of all the solutions to the given congruence. We want three positive solutions, so let's keep adding 15 until we get three positive solutions: -36, -21, -6, 9, 24, 39, …

So, **9, 24, and 39** are three positive integer solutions.
(There are other possible answers.)