MATH 245: QUIZ 4 SOLUTIONS

1)

Let c be the conjecture " $\sqrt{3}$ is irrational".
Let's use a proof by contradiction to prove c.

Assume $\neg c$, the conjecture " $\sqrt{3}$ is rational", is true.

Then, there exist relatively prime $a \in \mathbf{Z}$ and $b \in \mathbf{Z}\,(b \neq 0)$ such that $\sqrt{3} = \frac{a}{b}$. (The "relatively prime" condition ensures that the fraction is in lowest terms.)

$$\sqrt{3} = \frac{a}{b}$$

$$\Rightarrow \quad 3 = \frac{a^2}{b^2}$$

$$\Rightarrow 3b^2 = a^2$$

$b^2 \in \mathbf{Z}$, so 3 divides a^2 and is, therefore, a prime factor of a^2. Any prime factor of a^2 must also be a prime factor of a, so 3 must also divide a. That means that $\exists c \in \mathbf{Z}\colon a = 3c$.

$$\Rightarrow 3b^2 = (3c)^2$$

$$\Rightarrow 3b^2 = 9c^2$$

$$\Rightarrow \quad b^2 = 3c^2$$

$c^2 \in \mathbf{Z}$, so 3 divides b^2 and is, therefore, a prime factor of b^2. Any prime factor of b^2 must also be a prime factor of b, so 3 must also divide b.

We've shown that 3 must divide both a and b, which contradicts the relative primality of a and b.

Therefore, $\neg c$ is false, and c is true.

Another proof:

If $\sqrt{3}$ were rational, then it would be a rational root of the polynomial $x^2 - 3$. However, the Rational Roots Theorem suggests that the only possible rational roots of this polynomial are ± 1 and ± 3. $\sqrt{3}$ is not equal to any of those possibilities, so $\sqrt{3}$ must be irrational.

2)

a) Contrapositive: "If four real numbers are at least 25, then their sum is at least 100."

b) True. You didn't need to show this, but here's a proof of the contrapositive:

Let x_1, x_2, x_3, and x_4 be any four real numbers that are at least 25.
Then, their sum, $x_1 + x_2 + x_3 + x_4 \geq 25 + 25 + 25 + 25 = 100$.

3) The proof is given on p.191 in Rosen.

4)

Let $P(n)$ be "n cents of postage can be obtained by using 4-cent stamps and/or 7-cent stamps."

Basis Step:

$P(20)$ is true: 20 cents can be obtained by using five 4-cent stamps.

Inductive Step:

Let $n \in \mathbf{Z}$ ($n \geq 20$); in other words, let n be an arbitrary integer that is at least 20.

Inductive hypothesis: Assume $P(n)$ is true.

Show $P(n+1)$ is true.

Case 1: a 7-cent stamp can be used to help make n cents of postage

Using such a combination of stamps to make n cents of postage, we can then obtain n+1 cents by replacing a 7-cent stamp with two 4-cent stamps (value: 8 cents).

Case 2: only 4-cent stamps can be used to make n cents of postage

Since $n \geq 20$ by assumption, at least five 4-cent stamps must be used to make the n cents of postage. We can then obtain n+1 cents by replacing five 4-cent stamps (value: 20 cents) with three 7-cent stamps (value: 21 cents).

In either case, postage of n+1 cents can be obtained by using only 4-cent stamps and/or 7-cent stamps.

5) Let $P(n)$ be $a_n \leq \left(\frac{4}{3}\right)^n$.

By the nature of the recursive definition, our inductive step requires "lookups" of up to three "dominoes" back. So, we need three base cases.

<u>Basis Step:</u>

$P(0)$ is true: $a_0 = 1 \leq \left(\frac{4}{3}\right)^0 = 1$.

$P(1)$ is true: $a_1 = 1 \leq \left(\frac{4}{3}\right)^1 = \frac{4}{3}$.

$P(2)$ is true: $a_2 = 1 \leq \left(\frac{4}{3}\right)^2 = \frac{16}{9}$.

<u>Inductive Step:</u>

Let $n \in \mathbf{Z}$ $(n \geq 2)$; in other words, let n be an arbitrary integer that is at least 2.

Inductive hypothesis: Assume $P(0), P(1), P(2), \ldots, P(n)$ are true.
i.e., Assume $\forall k \; (0 \leq k \leq n) \; P(k)$.

Show $P(n+1)$: $a_{n+1} \leq \left(\frac{4}{3}\right)^{n+1}$ is true.

$$a_{n+1} = a_{n-1} + a_{n-2}$$

$$a_{n+1} \leq \left(\frac{4}{3}\right)^{n-1} + \left(\frac{4}{3}\right)^{n-2} \quad \text{(by the inductive hypothesis)}$$

$$a_{n+1} \leq \left(\frac{4}{3}\right)^{n-2}\left(\frac{4}{3}+1\right) \quad \left(\text{factor out } \left(\frac{4}{3}\right)^{n-2}\right)$$

$$a_{n+1} \leq \left(\frac{4}{3}\right)^{n-2}\left(\frac{7}{3}\right)$$

Remember, I want to show that $a_{n+1} \leq \left(\frac{4}{3}\right)^{n+1}$ is true.

Let's show that $\frac{7}{3} \leq \left(\frac{4}{3}\right)^3$:

$$\left(\frac{4}{3}\right)^3 = \frac{64}{27}, \text{ while } \frac{7}{3} = \frac{63}{27}, \text{ so } \frac{7}{3} \leq \left(\frac{4}{3}\right)^3.$$

$$a_{n+1} \leq \left(\frac{4}{3}\right)^{n-2}\left(\frac{7}{3}\right)$$

$$a_{n+1} \leq \left(\frac{4}{3}\right)^{n-2}\left(\frac{4}{3}\right)^3$$

$$a_{n+1} \leq \left(\frac{4}{3}\right)^{n+1}$$