

## MATH 245: QUIZ 4 SOLUTIONS

1)

Let  $c$  be the conjecture " $\sqrt{3}$  is irrational".

Let's use a proof by contradiction to prove  $c$ .

Assume  $\neg c$ , the conjecture " $\sqrt{3}$  is rational", is true.

Then, there exist relatively prime  $a \in \mathbf{Z}$  and  $b \in \mathbf{Z} (b \neq 0)$  such that

$\sqrt{3} = \frac{a}{b}$ . (The "relatively prime" condition ensures that the fraction is in lowest terms.)

$$\begin{aligned}\sqrt{3} &= \frac{a}{b} \\ \Rightarrow 3 &= \frac{a^2}{b^2} \\ \Rightarrow 3b^2 &= a^2\end{aligned}$$

$b^2 \in \mathbf{Z}$ , so 3 divides  $a^2$  and is, therefore, a prime factor of  $a^2$ . Any prime factor of  $a^2$  must also be a prime factor of  $a$ , so 3 must also divide  $a$ . That means that  $\exists c \in \mathbf{Z}: a = 3c$ .

$$\begin{aligned}\Rightarrow 3b^2 &= (3c)^2 \\ \Rightarrow 3b^2 &= 9c^2 \\ \Rightarrow b^2 &= 3c^2\end{aligned}$$

$c^2 \in \mathbf{Z}$ , so 3 divides  $b^2$  and is, therefore, a prime factor of  $b^2$ . Any prime factor of  $b^2$  must also be a prime factor of  $b$ , so 3 must also divide  $b$ .

We've shown that 3 must divide both  $a$  and  $b$ , which contradicts the relative primality of  $a$  and  $b$ .

Therefore,  $\neg c$  is false, and  $c$  is true.

Another proof:

If  $\sqrt{3}$  were rational, then it would be a rational root of the polynomial  $x^2 - 3$ . However, the Rational Roots Theorem suggests that the only possible rational roots of this polynomial are  $\pm 1$  and  $\pm 3$ .  $\sqrt{3}$  is not equal to any of those possibilities, so  $\sqrt{3}$  must be irrational.

2)

a) Contrapositive: "If four real numbers are at least 25, then their sum is at least 100."

b) True. You didn't need to show this, but here's a proof of the contrapositive:

Let  $x_1, x_2, x_3,$  and  $x_4$  be any four real numbers that are at least 25.  
Then, their sum,  $x_1 + x_2 + x_3 + x_4 \geq 25 + 25 + 25 + 25 = 100$ .

3) The proof is given on p.191 in Rosen.

4)

Let  $P(n)$  be " $n$  cents of postage can be obtained by using 4-cent stamps and/or 7-cent stamps."

Basis Step:

$P(20)$  is true: 20 cents can be obtained by using five 4-cent stamps.

Inductive Step:

Let  $n \in \mathbf{Z}$  ( $n \geq 20$ ); in other words, let  $n$  be an arbitrary integer that is at least 20.

Inductive hypothesis: Assume  $P(n)$  is true.

Show  $P(n+1)$  is true.

Case 1: a 7-cent stamp can be used to help make  $n$  cents of postage

Using such a combination of stamps to make  $n$  cents of postage, we can then obtain  $n+1$  cents by replacing a 7-cent stamp with two 4-cent stamps (value: 8 cents).

Case 2: only 4-cent stamps can be used to make  $n$  cents of postage

Since  $n \geq 20$  by assumption, at least five 4-cent stamps must be used to make the  $n$  cents of postage. We can then obtain  $n+1$  cents by replacing five 4-cent stamps (value: 20 cents) with three 7-cent stamps (value: 21 cents).

In either case, postage of  $n+1$  cents can be obtained by using only 4-cent stamps and/or 7-cent stamps.

5) Let  $P(n)$  be  $a_n \leq \left(\frac{4}{3}\right)^n$ .

By the nature of the recursive definition, our inductive step requires "lookups" of up to three "dominoes" back. So, we need three base cases.

Basis Step:

$$P(0) \text{ is true: } a_0 = 1 \leq \left(\frac{4}{3}\right)^0 = 1.$$

$$P(1) \text{ is true: } a_1 = 1 \leq \left(\frac{4}{3}\right)^1 = \frac{4}{3}.$$

$$P(2) \text{ is true: } a_2 = 1 \leq \left(\frac{4}{3}\right)^2 = \frac{16}{9}.$$

Inductive Step:

Let  $n \in \mathbf{Z}$  ( $n \geq 2$ ); in other words, let  $n$  be an arbitrary integer that is at least 2.

Inductive hypothesis: Assume  $P(0), P(1), P(2), \dots, P(n)$  are true.  
i.e., Assume  $\forall k$  ( $0 \leq k \leq n$ )  $P(k)$ .

Show  $P(n+1)$ :  $a_{n+1} \leq \left(\frac{4}{3}\right)^{n+1}$  is true.

$$a_{n+1} = a_{n-1} + a_{n-2}$$

$$a_{n+1} \leq \left(\frac{4}{3}\right)^{n-1} + \left(\frac{4}{3}\right)^{n-2} \quad (\text{by the inductive hypothesis})$$

$$a_{n+1} \leq \left(\frac{4}{3}\right)^{n-2} \left(\frac{4}{3} + 1\right) \quad \left(\text{factor out } \left(\frac{4}{3}\right)^{n-2}\right)$$

$$a_{n+1} \leq \left(\frac{4}{3}\right)^{n-2} \left(\frac{7}{3}\right)$$

Remember, I want to show that  $a_{n+1} \leq \left(\frac{4}{3}\right)^{n+1}$  is true.

Let's show that  $\frac{7}{3} \leq \left(\frac{4}{3}\right)^3$ :

$$\left(\frac{4}{3}\right)^3 = \frac{64}{27}, \text{ while } \frac{7}{3} = \frac{63}{27}, \text{ so } \frac{7}{3} \leq \left(\frac{4}{3}\right)^3.$$

$$a_{n+1} \leq \left(\frac{4}{3}\right)^{n-2} \left(\frac{7}{3}\right)$$

$$a_{n+1} \leq \left(\frac{4}{3}\right)^{n-2} \left(\frac{4}{3}\right)^3$$

$$a_{n+1} \leq \left(\frac{4}{3}\right)^{n+1}$$