MATH 245: QUIZ 6 SOLUTIONS

1) Let a_n = the number of bit strings of length n with at least one occurrence of "1111". Consider the following disjoint cases:

Case 1: $0\underbrace{\qquad}_{\text{length } n-1}$ a_{n-1} possibilities

Case 2: $10\underbrace{\qquad}_{\text{length } n-2}$ a_{n-2} possibilities

Case 3: $110\underbrace{\qquad}_{\text{length } n-3}$ a_{n-3} possibilities

Case 4: $1110\underbrace{\qquad}_{\text{length } n-4}$ a_{n-4} possibilities

Case 5: $1111\underbrace{\qquad}_{\text{length } n-4}$ 2^{n-4} possibilities, since the rest of the string can be any sequence of n-4 bits.

Answer: $a_n = a_{n-1} + a_{n-2} + a_{n-3} + a_{n-4} + 2^{n-4}$

2)

a) Let a_n = the number of decimal strings of length n with no occurrences of "000".

Case 1: $\dfrac{\text{9 possibilities}}{\text{can be 1 - 9}}\ \underbrace{\qquad\qquad}_{\text{length } n-1}$ $9a_{n-1}$ possibilities

Case 2: $\dfrac{1}{\text{"0"}}\ \dfrac{9}{\text{can be 1 - 9}}\ \underbrace{\qquad\qquad}_{\text{length } n-2}$ $9a_{n-2}$ possibilities

Case 3: $\dfrac{1}{\text{"0"}}\ \dfrac{1}{\text{"0"}}\ \dfrac{9}{\text{can be 1 - 9}}\ \underbrace{\qquad\qquad}_{\text{length } n-3}$ $9a_{n-3}$ possibilities

Answer: $a_n = 9a_{n-1} + 9a_{n-2} + 9a_{n-3}$

b)

There are 10^3 = 1000 decimal strings of length 3, and only one contains "000", namely "000", itself. Answer: 999.

c)

It might be a better idea to start off by counting the number of 4-length decimal strings that do contain "000".

Let A be the set of decimal strings of the form "000_". $|A| = 10$, since there are 10 ways to fill in "_".
Let B be the set of decimal strings of the form "_000". $|B| = 10$, since there are 10 ways to fill in "_".
There is exactly one string in both sets that is being double-counted, namely "0000".
So, $|A \cup B| = |A| + |B| - |A \cap B| = 10 + 10 - 1 = 19$.

There are a total of $10^4 = 10{,}000$ decimal strings of length 4, so the answer is 10,000 - 19 = 9981.

3)

$$a_n = 7a_{n-1} - 6a_{n-2}$$
$$a_n - 7a_{n-1} + 6a_{n-2} = 0$$
$$r^2 - 7r + 6 = 0$$
$$(r-6)(r-1) = 0$$

There are two roots: $r = 6$ and $r = 1$.

Solutions to the recurrence relation:

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$$
$$a_n = \alpha_1 (6)^n + \alpha_2 (1)^n$$
$$a_n = \alpha_1 (6)^n + \alpha_2$$

Use the initial conditions:

$$30 = \alpha_1 (6)^0 + \alpha_2$$
$$30 = \alpha_1 (1) + \alpha_2$$
$$30 = \alpha_1 + \alpha_2 \quad (*)$$

$$20 = \alpha_1 (6)^1 + \alpha_2$$
$$20 = 6\alpha_1 + \alpha_2 \quad (**)$$

Solving (*) and (**) for α_1 and α_2, we get α_1 = -2 and α_2 = 32.

Answer: $a_n = (-2)6^n + 32 \quad (n \geq 0)$

4)

$$
\begin{aligned}
a_n &= a_{n-1} + 3n + 2 \\
&= [a_{n-2} + 3(n-1) + 2] + 3n + 2 \\
&= a_{n-2} + 3(n-1) + 3n + 2 + 2 \\
&= [a_{n-3} + 3(n-2) + 2] + 3(n-1) + 3n + 2 + 2 \\
&= a_{n-3} + 3(n-2) + 3(n-1) + 3n + 2 + 2 + 2 \\
&\cdots \\
&= a_0 + 3(1) + 3(2) + \ldots + 3n + \underbrace{2 + 2 + \ldots + 2}_{n \text{ terms}} \\
&= 5 + 3(1 + 2 + \ldots + n) + 2n \\
&= 5 + 3\frac{n(n+1)}{2} + 2n
\end{aligned}
$$

(This can be simplified further.)

5)

Notation: "*E*" = "even". "-" means "any digit".
Let *A* be the set of decimal strings of the form "*EEE*--".
Let *B* be the set of decimal strings of the form "0----".
Let *C* be the set of decimal strings of the form "--*EEE*".

$|A|$: $\frac{5}{E}\,\frac{5}{E}\,\frac{5}{E}\,\frac{10}{\text{any}}\,\frac{10}{\text{any}}$ Product = 12,500

$|B|$: $\frac{1}{\text{"0"}}\,\frac{10}{\text{any}}\,\frac{10}{\text{any}}\,\frac{10}{\text{any}}\,\frac{10}{\text{any}}$ Product = 10,000

$|C|$: $\frac{10}{\text{any}}\,\frac{10}{\text{any}}\,\frac{5}{E}\,\frac{5}{E}\,\frac{5}{E}$ Product = 12,500 (symmetric to $|A|$ case)

$|A \cap B|$: $\frac{1}{\text{"0"}}\,\frac{5}{E}\,\frac{5}{E}\,\frac{10}{\text{any}}\,\frac{10}{\text{any}}$ Product = 2,500

$|B \cap C|$: $\frac{1}{\text{"0"}}\,\frac{10}{\text{any}}\,\frac{5}{E}\,\frac{5}{E}\,\frac{5}{E}$ Product = 1,250

$|A \cap C|$: $\frac{5}{E}\,\frac{5}{E}\,\frac{5}{E}\,\frac{5}{E}\,\frac{5}{E}$ Product = 3,125

$|A \cap B \cap C|$: $\frac{1}{\text{"0"}}\,\frac{5}{E}\,\frac{5}{E}\,\frac{5}{E}\,\frac{5}{E}$ Product = 625

So, $|A \cup B \cup C|$ = 12,500 + 10,000 + 12,500 - 2,500 - 1,250 - 3,125 + 625
Answer: 28,750.