FLOW LINES (Streamlines)

Author: Dr. Parvini

Determining the flow lines (also known as ***field lines, streamlines, integral curves***) of a vector field $\vec{F}$ usually amounts to solving a differential equation or a system of differential equations.

A curve ***C*** described by $\vec{r}(t)$is a flow line (integral curve) of vector field $\vec{F}$ if:

$$\boldsymbol{\vec{r}'(t) = \vec{F}(\vec{r}(t))}$$

[This means for each point $P = \vec{r}(t)$ of ***C***, the vector field $\vec{F}(P)$ is tangent to the flow line at P.]

Example –1: Determine the equation of flow lines or field lines of $\vec{F}(x, y) = \langle 1, y\rangle$.

We want $\vec{r}(t) = \langle x(t), y(t)\rangle$ such that:

$$\begin{aligned} \vec{r}'(t) &= \vec{F}(\vec{r}(t)) \\ \langle x'(t), y'(t)\rangle &= \langle 1, y(t)\rangle \end{aligned}$$

Equating the components of the two vectors yields:

$\begin{cases} x'(t) = 1 \\ y'(t) = y(t) \end{cases}$ Solve the first differential equation, $\int x'(t)dt = \int 1\, dt \;\; or\; x(t) = t + C_1$.

To solve the second differential equations:

$y'(t) - y(t) = 0$

$\frac{dy}{dt} = y(t)$ Using separation of variables: $\int \frac{dy}{y} = \int dt \xrightarrow{yields} \ln|y| = t + C_2$

$|y| = e^{t+C_2}$ or $y = \pm e^{t+C_2} = C_3 e^t$.

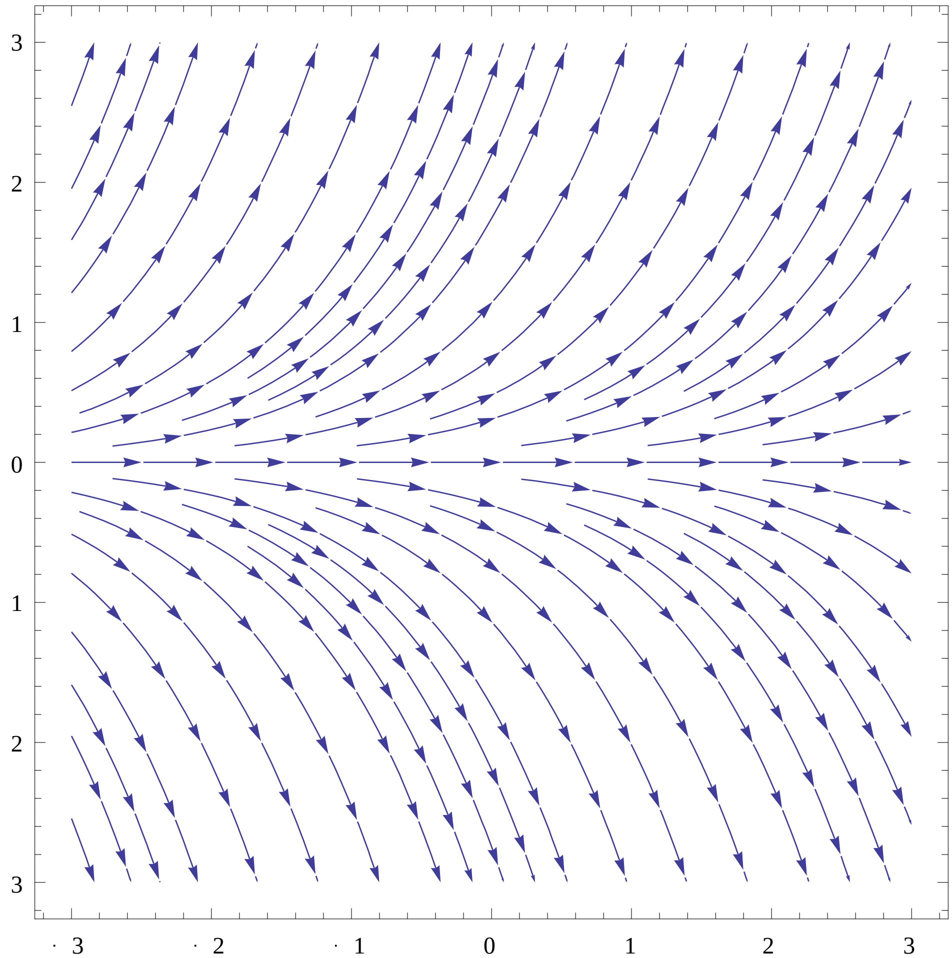
Substitute for *t* in $x(t) = t + C_1$ into the equation above: $y = C_3 e^{x-C_1} = C_3 e^{-C_1}. e^x = Ce^x$.

So that the flow lines of the vector field $\vec{F}(x, y) = \langle 1, y\rangle$ are graphs of the equation

$\boldsymbol{y = Ce^x}$ where C is a constant.

[Note: A curve that passes through $\vec{F}$ is called a ***line*** whether or not it is straight.]

A computer generated graph of the flow lines is given below:



Flow lines of vector filed $\vec{F}(x, y) = \langle 1, y \rangle$ are given by: $y = Ce^{x}$ where C is a constant.

Example – 2: Determine the flow lines of the vector field: $\vec{F}(x, y) = \langle -y, x \rangle$.

We want $\vec{r}(t) = \langle x(t), y(t) \rangle$ such that:

$$\begin{aligned} \vec{r}\,'(t) &= \vec{F}(\vec{r}(t)) \\ \langle x'(t), y'(t) \rangle &= \langle -y(t), x(t) \rangle \end{aligned}$$

Equating the components of the vectors above yields the following system of differential equations,

$$\begin{cases} x'(t) = -y(t) \\ y'(t) = x(t) \end{cases}$$

Differentiate the first equation with respect to t: $x''(t) = -y'(t) = x(t)$

Let $x(t) = a\, cost + b\, sint$ from which $x'(t) = -a\, sint\ + b\, cost$.

From the first differential equation, above in the system we get:

$y(t) = -x'(t) = -b\, cost\ + a\, sint$

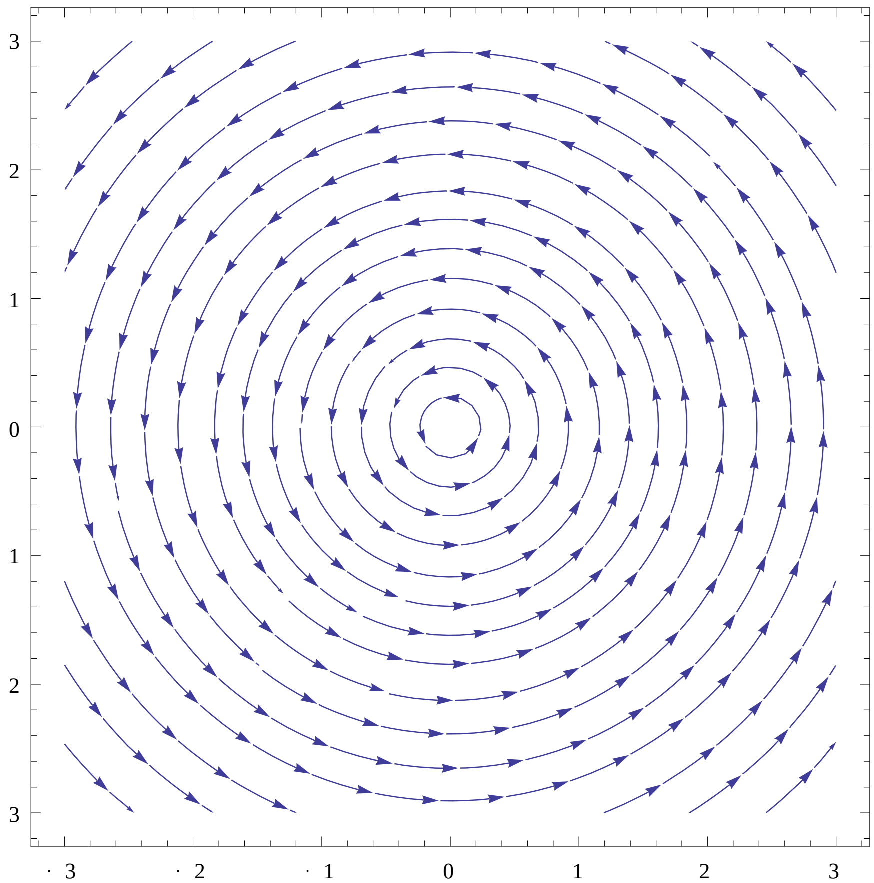
so that: $\boldsymbol{\vec{r}(t) = \langle a\, cost + b\, sint\, ,\ -b\, cost\ + a\, sint \rangle \quad 0 \leq t \leq 2\pi}$

If B=0, then: $\vec{r}(t) = \langle a\, cost\ ,\ a\, sint \rangle$ is the counterclockwise parameterization of a circle of radius $|a|$.

Otherwise, $|\vec{r}(t)| = |\langle a\, cost + b\, sint\, ,\ -b\, cost\ + a\, sint \rangle| = \sqrt{a^2 + b^2}$.

(details omitted) parameterizes a circle of radius $\sqrt{a^2 + b^2}$.

A computer generated graph of the flow lines is given below:



Flow lines of vector filed $\vec{\boldsymbol{F}}(\boldsymbol{x},\boldsymbol{y}) = \langle -\boldsymbol{y}, \boldsymbol{x}\rangle$ are given by the parametric equation:

$\vec{\boldsymbol{r}}(\boldsymbol{t}) = \langle \boldsymbol{a\,cost} + \boldsymbol{b\,sint}\,,\ -\boldsymbol{b\,cost}\ + \boldsymbol{a\,sint}\rangle \quad \mathbf{0} \le \boldsymbol{t} \le \mathbf{2}\boldsymbol{\pi}$.