QUIZ 1 (CHAPTER 14) - SOLUTIONS

MATH 252 – FALL 2007 – KUNIYUKI

SCORED OUT OF 125 POINTS ⇒ MULTIPLIED BY 0.84 ⇒ 105% POSSIBLE

Clearly mark vectors, as we have done in class. I will use boldface, but you don't! When describing vectors, you may use either $\langle \ \rangle$ or "$\mathbf{i}-\mathbf{j}-\mathbf{k}$" notation.

Assume we are in our usual 2- and 3-dimensional Cartesian coordinate systems. Give exact answers, unless otherwise specified.

1) Assume that a_1, a_2, p, and q are real numbers.
   Prove that, if $\mathbf{a} = \langle a_1, a_2 \rangle$, then $(p+q)\mathbf{a} = p\mathbf{a} + q\mathbf{a}$. Show <u>all</u> steps! (10 points)

Let $\mathbf{a} = \langle a_1, a_2 \rangle$.

$$\begin{aligned}
(p+q)\mathbf{a} &= (p+q)\langle a_1, a_2 \rangle \\
&= \langle (p+q)a_1,\ (p+q)a_2 \rangle \\
&= \langle pa_1 + qa_1,\ pa_2 + qa_2 \rangle \\
&= \langle pa_1, pa_2 \rangle + \langle qa_1, qa_2 \rangle \\
&= p\langle a_1, a_2 \rangle + q\langle a_1, a_2 \rangle \\
&= p\mathbf{a} + q\mathbf{a}
\end{aligned}$$

Q.E.D.

<u>Note</u>: Many people who scored 4 points did the following:

$$\begin{aligned}
(p+q)\mathbf{a} &= (p+q)\langle a_1, a_2 \rangle \\
&= p\langle a_1, a_2 \rangle + q\langle a_1, a_2 \rangle \quad \leftarrow \text{You're using the property you're trying to prove! That's circular reasoning!} \\
&= p\mathbf{a} + q\mathbf{a}
\end{aligned}$$

2) Write an inequality in x, y, and/or z whose graph in our usual three-dimensional xyz-coordinate system consists of the sphere of radius 4 centered at the origin and all points inside that sphere. (4 points)

The equation of the sphere of radius 4 centered at the origin: $x^2 + y^2 + z^2 = 16$.

We also want all points inside that sphere, so our inequality is: $\boxed{x^2 + y^2 + z^2 \le 16}$.

Another approach: We want all points whose distance from the origin is at most 4.

3) Find all real values of c such that the vectors $c\mathbf{i}+10\mathbf{j}+c\mathbf{k}$ and $c\mathbf{i}-2\mathbf{j}-\mathbf{k}$ are orthogonal. (8 points)

The vectors are orthogonal $\Leftrightarrow$ Their dot product is 0.

$$\begin{aligned}\left(c\mathbf{i}+10\mathbf{j}+c\mathbf{k}\right)\bullet\left(c\mathbf{i}-2\mathbf{j}-\mathbf{k}\right)&=\langle c,10,c\rangle\bullet\langle c,-2,-1\rangle\\&=c^2-20-c\\&=c^2-c-20\end{aligned}$$

Find the real zeros:

$$\begin{gathered}c^2-c-20=0\\(c+4)(c-5)=0\\\boxed{c=-4\quad\text{or}\quad c=5}\end{gathered}$$

4) Assume that $\mathbf{a}$ and $\mathbf{b}$ are vectors in V_n, where n is some natural number. Using entirely mathematical notation (i.e., don't use words) …
(8 points; 4 points each)

a) Write the Cauchy-Schwarz Inequality.

$$\boxed{\left|\mathbf{a}\bullet\mathbf{b}\right|\le\left\|\mathbf{a}\right\|\left\|\mathbf{b}\right\|}$$

b) Write the Triangle Inequality.

$$\boxed{\left\|\mathbf{a}+\mathbf{b}\right\|\le\left\|\mathbf{a}\right\|+\left\|\mathbf{b}\right\|}$$

5) Let $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ be vectors in V_3. (4 points total; 2 points each)

a) $(\mathbf{a}\bullet\mathbf{b})\mathbf{c}$ is … (Box in one:)

a scalar $\qquad$ $\boxed{\textbf{a vector}}$ $\qquad$ neither, or undefined

$\mathbf{a}\bullet\mathbf{b}$ is a scalar, and a scalar times a vector is a vector.
Think: Scalar multiplication.

b) $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ is … (Box in one:)

a scalar $\qquad$ $\boxed{\textbf{a vector}}$ $\qquad$ neither, or undefined

In fact, this is a Triple Vector Product. $\mathbf{b}\times\mathbf{c}$ is a vector, and $\mathbf{a}$ crossed with it is a vector.

6) Assume that **a**, **b**, and **c** are three nonzero vectors in V_3 such that $\mathbf{a}\times\mathbf{b}=\mathbf{a}\times\mathbf{c}$. Which of the following must be true? Box in one: (3 points)

i) The vector **a** and the vector $\mathbf{b}-\mathbf{c}$ are parallel.

ii) The vector **a** and the vector $\mathbf{b}-\mathbf{c}$ are perpendicular (or orthogonal).

iii) $\mathbf{b}=\mathbf{c}$.

$$\begin{aligned}\mathbf{a}\times\mathbf{b}=\mathbf{a}\times\mathbf{c} &\quad\Leftrightarrow\\ \mathbf{a}\times\mathbf{b}-\mathbf{a}\times\mathbf{c}=\mathbf{0} &\quad\Leftrightarrow\\ \mathbf{a}\times(\mathbf{b}-\mathbf{c})=\mathbf{0} &\quad\Leftrightarrow\\ \mathbf{a}\parallel(\mathbf{b}-\mathbf{c})\end{aligned}$$

7) The line l passes through the points $P(-7,2,0)$ and $Q(4,-1,5)$. (10 points total)

a) Find parametric equations for l.

First, find a direction vector for l:

$$\begin{aligned}\overline{PQ}&=\langle 4-(-7),-1-2,5-0\rangle\\ &=\langle 11,-3,5\rangle\end{aligned}$$

Use this direction vector together with one of the given points (say P) to obtain parametric equations for l:

$$\boxed{\begin{cases}x=-7+11t\\ y=\ \ 2-\ 3t\ , & t \text{ in } \mathbf{R}\\ z=\qquad 5t\end{cases}}$$

b) Find symmetric equations for l.

Solve the three equations in a) for t and equate the resulting expressions for t.

$$\boxed{\frac{x+7}{11}=\frac{y-2}{-3}=\frac{z}{5}}$$

8) Consider the following two lines:

$$l_1 : \begin{cases} x = \ \ 3 + \ t \\ y = \ \ 1 - 2t \\ z = -5 + 3t \end{cases} \quad \text{and} \quad l_2 : \begin{cases} x = -2 - 3u \\ y = \ \ 3 + 4u \\ z = \ \ 8 - 2u \end{cases} \qquad (t, u \in \mathbf{R})$$

(35 points total)

a) Find the point of intersection between the two lines.

Equate the expressions for corresponding coordinates, and solve the system:

$$\begin{cases} 3 + t = -2 - 3u \\ 1 - 2t = 3 + 4u \\ -5 + 3t = 8 - 2u \end{cases} \Leftrightarrow$$

$$\begin{cases} t + 3u = -5 & (\text{Eq.}1) \\ -2t - 4u = 2 & (\text{Eq.}2) \\ 3t + 2u = 13 & (\text{Eq.}3) \end{cases}$$

Solve the subsystem with, say, the first two equations:

$$\begin{cases} t + 3u = -5 & (\text{Eq.}1) \\ -2t - 4u = 2 & (\text{Eq.}2) \end{cases}$$

The unique solution is: $(t = 7, u = -4)$.

Verify that $(t = 7, u = -4)$ satisfies Eq.3:

$$\begin{aligned} 3t + 2u &= 13, \quad (t = 7, u = -4) \quad \Rightarrow \\ 3(7) + 2(-4) &= 13 \\ 13 &= 13 \quad (\text{Checks out.}) \end{aligned}$$

Therefore, the two given lines intersect at the point for which $(t = 7, u = -4)$.

Find the intersection point:

We will substitute $t = 7$ into the equations for l_1.
(Alternately, we could substitute $u = -4$ into the equations for l_2.)

$$\begin{cases} x = 3 + (7) = 10 \\ y = 1 - 2(7) = -13 \\ z = -5 + 3(7) = 16 \end{cases}$$

The intersection point is: $\boxed{(10, -13, 16).}$

b) Find either one of the two supplementary angles between the given lines.
Reminder:

$$l_1 : \begin{cases} x = 3 + t \\ y = 1 - 2t \\ z = -5 + 3t \end{cases} \quad \text{and} \quad l_2 : \begin{cases} x = -2 - 3u \\ y = 3 + 4u \\ z = 8 - 2u \end{cases} \qquad (t, u \in \mathbf{R})$$

Round off your answer to the nearest tenth of a degree.

A direction vector for l_1 is given by $\mathbf{a} = \langle 1, -2, 3 \rangle$.
A direction vector for l_2 is given by $\mathbf{b} = \langle -3, 4, -2 \rangle$.

Ingredients for our angle formula:

$$\begin{aligned} \mathbf{a} \bullet \mathbf{b} &= \langle 1, -2, 3 \rangle \bullet \langle -3, 4, -2 \rangle \\ &= (1)(-3) + (-2)(4) + (3)(-2) \quad (\leftarrow \text{Maybe easier to skip.}) \\ &= -3 - 8 - 6 \\ &= -17 \end{aligned}$$

$$\|\mathbf{a}\| = \|\langle 1, -2, 3 \rangle\| = \sqrt{(1)^2 + (-2)^2 + (3)^2} = \sqrt{14}$$

$$\|\mathbf{b}\| = \|\langle -3, 4, -2 \rangle\| = \sqrt{(-3)^2 + (4)^2 + (-2)^2} = \sqrt{29}$$

Find an angle:

$$\theta = \cos^{-1}\left(\frac{\mathbf{a} \bullet \mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|}\right)$$

$$= \cos^{-1}\left(\frac{-17}{\sqrt{14}\sqrt{29}}\right) \quad \left[\underline{\text{Note}}:\ = \cos^{-1}\left(-\frac{17}{\sqrt{406}}\right) \approx \cos^{-1}(-0.843696)\right]$$

$$\approx \boxed{147.5^\circ}$$

Note 1: Because either of the direction vectors we found could be reversed, the supplementary angle, about 32.5°, would also have been acceptable.

Note 2: Some books require that arccosine values be written in radians, but we won't worry about that.

c) Find an equation (in x, y, and z) of the plane that contains the two given lines. Reminder:

$$l_1 : \begin{cases} x = 3 + t \\ y = 1 - 2t \\ z = -5 + 3t \end{cases} \quad \text{and} \quad l_2 : \begin{cases} x = -2 - 3u \\ y = 3 + 4u \\ z = 8 - 2u \end{cases} \quad (t, u \in \mathbf{R})$$

A direction vector for l_1 is given by $\mathbf{a} = \langle 1, -2, 3 \rangle$.
A direction vector for l_2 is given by $\mathbf{b} = \langle -3, 4, -2 \rangle$.

Since **a** and **b** are nonparallel vectors, we can obtain a normal vector **n** for our plane as follows:

$$\begin{aligned}
\mathbf{a} \times \mathbf{b} &= \langle 1, -2, 3 \rangle \times \langle -3, 4, -2 \rangle \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 3 \\ -3 & 4 & -2 \end{vmatrix} \\
&= \begin{vmatrix} -2 & 3 \\ 4 & -2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 3 \\ -3 & -2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & -2 \\ -3 & 4 \end{vmatrix}\mathbf{k} \\
&= (4 - 12)\mathbf{i} - (-2 - (-9))\mathbf{j} + (4 - 6)\mathbf{k} \\
&= -8\mathbf{i} - 7\mathbf{j} - 2\mathbf{k} \\
&= \langle -8, -7, -2 \rangle
\end{aligned}$$

For simplicity, we can use the opposite vector as our **n**: $\mathbf{n} = \langle 8, 7, 2 \rangle$.

We need a point on the desired plane. Three immediate choices are:

$(3, 1, -5)$, which is on line l_1,
$(-2, 3, 8)$, which is on line l_2, and
$(10, -13, 16)$, which is the intersection point we found in part a).

Let's use $\mathbf{n} = \langle 8, 7, 2 \rangle$ as our normal and $(3, 1, -5)$, say, as our point.

Standard form for an equation of the plane:

$$\boxed{8(x-3)+7(y-1)+2(z-(-5))=0}$$

General form for an equation of the plane:

$$\boxed{8x+7y+2z-21=0}$$

9) Consider the point $P(7, 2, -1)$ and the plane $4x-3y+2z+60=0$. Distance is measured in meters. (16 points total)

a) At what point does the given plane intersect the *x*-axis? (We will call this point Q.)

Along the *x*-axis, $y=0$ and $z=0$, so we substitute $y=0$ and $z=0$ in the given equation and solve for *x*:

$$\begin{aligned} 4x-3y+2z+60=0, \quad y=0, \quad z=0 \quad &\Rightarrow \\ 4x-3(0)+2(0)+60&=0 \\ 4x+60&=0 \\ x&=-15 \end{aligned}$$

The desired point is: $\boxed{Q(-15, 0, 0).}$

b) Find a normal vector for the given plane. (We will call this vector **n**.)

$$\boxed{\mathbf{n} = \langle 4, -3, 2 \rangle.}$$

c) If we let the vector $\mathbf{p} = \overrightarrow{QP}$, then the distance between the given point P and the given plane equals: $\left|\text{comp}_{\mathbf{n}}\mathbf{p}\right|$. Use the component formula to find $\left|\text{comp}_{\mathbf{n}}\mathbf{p}\right|$. Round it off to the nearest tenth of a meter.

Find the **p** vector:

$$\begin{aligned}\mathbf{p} &= \overrightarrow{QP} \\ &= \langle 7-(-15), 2-0, -1-0\rangle \\ &= \langle 22, 2, -1\rangle\end{aligned}$$

The desired distance is:

$$\begin{aligned}\left|\text{comp}_{\mathbf{n}}\mathbf{p}\right| &= \frac{\left|\mathbf{p}\bullet\mathbf{n}\right|}{\left\|\mathbf{n}\right\|} \\ &= \frac{\left|\langle 22, 2, -1\rangle \bullet \langle 4, -3, 2\rangle\right|}{\left\|\langle 4, -3, 2\rangle\right\|} \\ &= \frac{\left|88-6-2\right|}{\sqrt{(4)^2+(-3)^2+(2)^2}} \\ &= \frac{80}{\sqrt{29}} \\ &\approx \boxed{14.9 \text{ meters}}\end{aligned}$$

Observe that we would have gotten the same answer from the shortcut formula given in Section 14.5 for a point (x_0, y_0, z_0) and a plane $ax+by+cz+d=0$:

$$\begin{aligned}h &= \frac{\left|ax_0+by_0+cz_0+d\right|}{\sqrt{a^2+b^2+c^2}} \\ &= \frac{\left|4(7)-3(2)+2(-1)+60\right|}{\sqrt{(4)^2+(-3)^2+(2)^2}} \\ &= \frac{80}{\sqrt{29}} \\ &\approx 14.9 \text{ meters}\end{aligned}$$

10) Matching. (12 points total)

Fill in each blank below with one of the following:

A. An Ellipsoid
B. A Hyperboloid of One Sheet
C. A Hyperboloid of Two Sheets
D. A Cone
E. An Elliptic Paraboloid
F. A Hyperbolic Paraboloid

I. The graph of $\frac{1}{2}x^2 - 3y^2 - z^2 = 5$ is __**C**__.

(Think: $x^2 - y^2 - z^2 = 1$.)

II. The graph of $x^2 + 7y^2 - z = 0$ is __**E**__.

(Think: $x^2 + y^2 - z = 0$, or $z = x^2 + y^2$.)

III. The graph of $4x^2 - 9y^2 + z^2 = 0$ is __**D**__.

(Think: $x^2 - y^2 + z^2 = 0$, or $y^2 = x^2 + z^2$.)

IV. The graph of $4x^2 - y^2 + 11z^2 = 7$ is __**B**__.

(Think: $x^2 - y^2 + z^2 = 1$.)

11) Consider the graph of $4x^2 - y^2 + 11z^2 = 7$. This was in Problem 10, part IV. Assume that k takes the place of real constants. (12 points total)

The graph is a hyperboloid of one sheet. For simplicity, consider: $x^2 - y^2 + z^2 = 1$

a) The axis of the graph is the … (Box in one:)

x-axis $\boxed{y\text{-axis}}$ z-axis

Observe that y is the "odd man out" in $x^2 - y^2 + z^2 = 1$.

b) The conic traces of the graph in the planes $x = k$ are … (Box in one:)

Ellipses $\boxed{\text{Hyperbolas}}$ Parabolas

Let k be a real number such that $|k| > 1$. The trace of the graph of $x^2 - y^2 + z^2 = 1$ in the plane $x = k$ is given by:

$$\begin{aligned} k^2 - y^2 + z^2 &= 1, & x &= k \\ k^2 - 1 &= y^2 - z^2, & x &= k \\ y^2 - z^2 &= \underbrace{k^2 - 1}_{>0}, & x &= k \end{aligned}$$

We obtain hyperbolas.

c) The conic traces of the graph in the planes $y = k$ are … (Box in one:)

$\boxed{\text{Ellipses}}$ Hyperbolas Parabolas

Let k be a real number. The trace of the graph of $x^2 - y^2 + z^2 = 1$ in the plane $y = k$ is given by:

$$\begin{aligned} x^2 - k^2 + z^2 &= 1, & y &= k \\ x^2 + z^2 &= \underbrace{1 + k^2}_{>0}, & y &= k \end{aligned}$$

This is a circle, but we have ellipses for the traces of the graph of the original equation because of the deformations produced by the coefficients.

The ellipse family of traces makes sense, since planes of the form $y = k$ are perpendicular to the axis of the graph of $x^2 - y^2 + z^2 = 1$, which is a hyperboloid of one sheet.

d) The conic traces of the graph in the planes $z = k$ are … (Box in one:)

Ellipses $\boxed{\text{Hyperbolas}}$ Parabolas

Let k be a real number such that $|k| > 1$. The trace of the graph of $x^2 - y^2 + z^2 = 1$ in the plane $z = k$ is given by:

$$\begin{aligned} x^2 - y^2 + k^2 &= 1, & z &= k \\ k^2 - 1 &= y^2 - x^2, & z &= k \\ y^2 - x^2 &= \underbrace{k^2 - 1}_{>0}, & z &= k \end{aligned}$$

We obtain hyperbolas.

12) Find an equation (in x, y, and z) of the surface obtained by revolving the graph of $4y^2 + 25z^2 = 1$ (in the yz-plane) about the z-axis. (3 points)

Since y is the "non-axis" variable in the equation above, and x is the "missing variable" …

We replace y^2 with $\left(x^2 + y^2\right)$. We don't "touch z."

$$4\left(x^2 + y^2\right) + 25z^2 = 1$$
$$\boxed{4x^2 + 4y^2 + 25z^2 = 1}$$

We are taking an ellipse in the yz-plane, and we are using it to generate an ellipsoid.