QUIZ 1 (CHAPTER 14) - SOLUTIONS

MATH 252 – FALL 2008 – KUNIYUKI

SCORED OUT OF 125 POINTS $\Rightarrow$ MULTIPLIED BY 0.84 $\Rightarrow$ 105% POSSIBLE

Clearly mark vectors, as we have done in class. I will use boldface, but you don't!
When describing vectors, you may use either $\langle\ \rangle$ or "$\mathbf{i}-\mathbf{j}-\mathbf{k}$" notation.
Assume we are in our usual 2- and 3-dimensional Cartesian coordinate systems.
Give exact answers, unless otherwise specified.

1) Consider the points $P(1,3)$ and $Q(5,8)$. Find the vector in V_2 that has the same direction as the vector [corresponding to] $\overrightarrow{PQ}$ and has length 100. (8 points)

Let **a** be the vector corresponding to $\overrightarrow{PQ}$:

$$\mathbf{a} = \langle 5-1, 8-3\rangle$$
$$= \langle 4, 5\rangle$$

Find the magnitude (or length) of **a**:

$$\|\mathbf{a}\| = \|\langle 4,5\rangle\| = \sqrt{(4)^2+(5)^2} = \sqrt{41}$$

Find the unit vector **u** in V_2 that has the same direction as **a**:

$$\mathbf{u} = \frac{\mathbf{a}}{\|\mathbf{a}\|} = \frac{\langle 4,5\rangle}{\sqrt{41}} = \left\langle \frac{4}{\sqrt{41}}, \frac{5}{\sqrt{41}}\right\rangle \text{ or } \left\langle \frac{4\sqrt{41}}{41}, \frac{5\sqrt{41}}{41}\right\rangle$$

Multiply **u** by 100 to get the desired vector of magnitude 100:

$$100\mathbf{u} = 100\left\langle \frac{4\sqrt{41}}{41}, \frac{5\sqrt{41}}{41}\right\rangle = \boxed{\left\langle \frac{400\sqrt{41}}{41}, \frac{500\sqrt{41}}{41}\right\rangle}$$

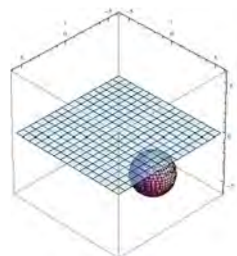
2) Write an equation (in *x*, *y*, and *z*) of the sphere with center $(0, 4, -2)$ that is tangent to the *xy*-plane in our usual three-dimensional *xyz*-coordinate system. (4 points)

The center of the sphere has *z*-coordinate -2, so it is two units below the *xy*-plane, and the radius of the sphere is 2.

We use the template: $(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$.

$$(x-0)^2 + (y-4)^2 + (z-(-2))^2 = (2)^2$$

$$\boxed{x^2 + (y-4)^2 + (z+2)^2 = 4}$$



3) Assume that **a**, **b**, and **c** are nonzero vectors in V_3.

Prove: $\text{comp}_{\mathbf{c}}(\mathbf{a}+\mathbf{b}) = \text{comp}_{\mathbf{c}}\mathbf{a} + \text{comp}_{\mathbf{c}}\mathbf{b}$.

You may use the comp formula and the basic dot product properties listed in the book without proof. (Writing $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and so forth will not help here.)

(6 points)

$$\begin{aligned}
\text{comp}_{\mathbf{c}}(\mathbf{a}+\mathbf{b}) &= \frac{(\mathbf{a}+\mathbf{b})\bullet\mathbf{c}}{\|\mathbf{c}\|} \\
&= \frac{\mathbf{a}\bullet\mathbf{c} + \mathbf{b}\bullet\mathbf{c}}{\|\mathbf{c}\|} \\
&= \frac{\mathbf{a}\bullet\mathbf{c}}{\|\mathbf{c}\|} + \frac{\mathbf{b}\bullet\mathbf{c}}{\|\mathbf{c}\|} \\
&= \text{comp}_{\mathbf{c}}\mathbf{a} + \text{comp}_{\mathbf{c}}\mathbf{b}
\end{aligned}$$

4) Write the Cauchy-Schwarz Inequality. Let **a** and **b** be vectors in V_n, where n is some natural number. (4 points)

$$\boxed{|\mathbf{a}\bullet\mathbf{b}| \le \|\mathbf{a}\|\,\|\mathbf{b}\|}$$

5) You do <u>not</u> have to show work for this problem. There are many possible answers for part a) and for part b). (6 points total; 3 points each)

a) Assuming $\mathbf{v} = \langle 2, 3 \rangle$ in V_2, find a nonzero vector $\mathbf{w}$ in V_2 that is orthogonal to $\mathbf{v}$.

Find a nonzero vector $\mathbf{w}$ in V_2 such that $\mathbf{v} \bullet \mathbf{w} = 0$.

Sample answer: $\boxed{\langle 3, -2 \rangle.}$

b) Assuming $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$ in V_3, find a nonzero vector $\mathbf{b}$ in V_3 such that $\mathbf{a} \times \mathbf{b} = \mathbf{0}$.

Find a nonzero vector $\mathbf{b}$ in V_3 that is parallel to $\mathbf{a}$.
We can choose any nonzero scalar multiple of $\mathbf{a}$, even $\mathbf{a}$ itself.

Sample answer: $\boxed{\mathbf{a}\text{, or } \langle 3, 4, 5 \rangle.}$

6) Let **a**, **b**, and **c** be vectors in V_3. (4 points total; 2 points each)

a) $(\mathbf{a} \times \mathbf{b}) \bullet \mathbf{c}$ is … (Box in one:)

$\boxed{\textbf{a scalar}}$ a vector neither, or undefined

$\mathbf{a} \times \mathbf{b}$ is a vector in V_3, and the dot product of a vector in V_3 and a vector in V_3 is a scalar. We call $(\mathbf{a} \times \mathbf{b}) \bullet \mathbf{c}$ a triple scalar product ("TSP").

b) $\mathbf{a} \times (\mathbf{b} \bullet \mathbf{c})$ is … (Box in one:)

a scalar a vector $\boxed{\textbf{neither, or undefined}}$

$\mathbf{b} \bullet \mathbf{c}$ is a scalar, and the cross product of a vector and a scalar is undefined.

7) The line l passes through the points $P(4,1,-3)$ and $Q(6,-1,-8)$.
(10 points total)

a) Find parametric equations for l.

First, find a direction vector for l:

$$\begin{aligned}\overrightarrow{PQ} &= \langle 6-4, -1-1, -8-(-3)\rangle \\ &= \langle 2,-2,-5\rangle\end{aligned}$$

Use this direction vector together with one of the given points (say P) to obtain parametric equations for l:

$$\boxed{\begin{cases} x = 4+2t \\ y = 1-2t\,, \quad t \text{ in } \mathbf{R} \\ z = -3-5t \end{cases}}$$

b) Find symmetric equations for l.

Solve the three equations in a) for t and equate the resulting expressions for t.

$$\boxed{\frac{x-4}{2} = \frac{y-1}{-2} = \frac{z+3}{-5}}$$

8) Consider the following two lines:

$$l_1 : \begin{cases} x = 13+4t \\ y = 5+2t \\ z = -2+3t \end{cases} \quad \text{and} \quad l_2 : \begin{cases} x = -5-3u \\ y = -9+u \\ z = -8-6u \end{cases} \qquad (t, u \in \mathbf{R})$$

(31 points total)

a) Find the point of intersection between the two lines.

Equate the expressions for corresponding coordinates, and solve the system:

$$\begin{cases} 13+4t = -5-3u \\ 5+2t = -9+u \\ -2+3t = -8-6u \end{cases} \Leftrightarrow$$

$$\begin{cases} 4t + 3u = -18 & (\text{Eq.}1) \\ 2t - \ \ u = -14 & (\text{Eq.}2) \\ 3t + 6u = \ -6 & (\text{Eq.}3) \end{cases}$$

Solve the subsystem with, say, the first two equations:

$$\begin{cases} 4t + 3u = -18 & (\text{Eq.}1) \\ 2t - \ \ u = -14 & (\text{Eq.}2) \end{cases}$$

The unique solution is: $(t = -6, u = 2)$.

Verify that $(t = -6, u = 2)$ satisfies Eq.3:

$$\begin{aligned} 3t + 6u &= -6, \quad (t = -6, u = 2) \quad \Rightarrow \\ 3(-6) + 6(2) &= -6 \\ -6 &= -6 \quad (\text{Checks out.}) \end{aligned}$$

Therefore, the two given lines intersect at the point for which $(t = -6, u = 2)$.

Find the intersection point:

We will substitute $u = 2$ into the equations for l_2.
(Alternately, we could substitute $t = -6$ into the equations for l_1.)

$$\begin{cases} x = -5 - 3(2) = -11 \\ y = -9 + \ \ (2) = -7 \\ z = -8 - 6(2) = -20 \end{cases}$$

The intersection point is: $\boxed{(-11, -7, -20).}$

b) Find either one of the two supplementary angles between the given lines. Reminder:

$$l_1 : \begin{cases} x = 13 + 4t \\ y = 5 + 2t \\ z = -2 + 3t \end{cases} \quad \text{and} \quad l_2 : \begin{cases} x = -5 - 3u \\ y = -9 + u \\ z = -8 - 6u \end{cases} \qquad (t, u \in \mathbf{R})$$

Round off your answer to the nearest tenth of a degree.

A direction vector for l_1 is given by $\mathbf{a} = \langle 4, 2, 3 \rangle$.
A direction vector for l_2 is given by $\mathbf{b} = \langle -3, 1, -6 \rangle$.

Ingredients for our angle formula:

$$\begin{aligned} \mathbf{a} \bullet \mathbf{b} &= \langle 4, 2, 3 \rangle \bullet \langle -3, 1, -6 \rangle \\ &= (4)(-3) + (2)(1) + (3)(-6) \quad (\leftarrow \text{Maybe easier to skip.}) \\ &= -12 + 2 - 18 \\ &= -28 \end{aligned}$$

$$\|\mathbf{a}\| = \|\langle 4, 2, 3 \rangle\| = \sqrt{(4)^2 + (2)^2 + (3)^2} = \sqrt{29}$$

$$\|\mathbf{b}\| = \|\langle -3, 1, -6 \rangle\| = \sqrt{(-3)^2 + (1)^2 + (-6)^2} = \sqrt{46}$$

Find an angle:

$$\begin{aligned} \theta &= \cos^{-1}\left(\frac{\mathbf{a} \bullet \mathbf{b}}{\|\mathbf{a}\| \|\mathbf{b}\|} \right) \\ &= \cos^{-1}\left(\frac{-28}{\sqrt{29}\sqrt{46}} \right) \quad \left[\underline{\text{Note}}:\ = \cos^{-1}\left(-\frac{28}{\sqrt{1334}} \right) \approx \cos^{-1}(-0.76662) \right] \\ &\approx \boxed{140.1^\circ} \end{aligned}$$

Note 1: Because either of the direction vectors we found could be reversed, the supplementary angle, about 39.9°, would also have been acceptable.

Note 2: Some books require that arccosine values be written in radians, but we won't worry about that.

c) At what point does the line l_2 intersect the *yz*-plane? Reminder:

$$l_2 : \begin{cases} x = -5 - 3u \\ y = -9 + u \quad (u \in \mathbf{R}) \\ z = -8 - 6u \end{cases}$$

We have to find the point on l_2 where $x = 0$.

Using the parametric equations for l_2:

Solve for t:

$$\begin{aligned} x &= 0 \\ -5 - 3u &= 0 \\ u &= -\frac{5}{3} \end{aligned}$$

We then have:

$$\begin{cases} x = 0 \\ y = -9 + u = -9 + \left(-\frac{5}{3}\right) = -9 - \frac{5}{3} = -\frac{32}{3} \\ z = -8 - 6u = -8 - 6\left(-\frac{5}{3}\right) = -8 + 10 = 2 \end{cases}$$

The desired point is: $\boxed{\left(0, -\frac{32}{3}, 2\right), \text{ or } \left(0, -10\frac{2}{3}, 2\right)}$

9) Consider the points $P(0, 4, 2)$, $Q(-1, 7, -3)$, and $R(2, 1, -1)$.
Distance is measured in meters. (20 points total)

a) Find an equation (in *x*, *y*, and *z*) of the plane containing the three points *P*, *Q*, and *R*.

$$\begin{aligned} \overrightarrow{PQ} &= \langle -1 - 0, 7 - 4, -3 - 2 \rangle \\ &= \langle -1, 3, -5 \rangle \end{aligned} \qquad \begin{aligned} \overrightarrow{PR} &= \langle 2 - 0, 1 - 4, -1 - 2 \rangle \\ &= \langle 2, -3, -3 \rangle \end{aligned}$$

Since $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are nonparallel vectors, we can obtain a normal vector **n** for the desired plane as follows:

$$\begin{aligned}
\overrightarrow{PQ} \times \overrightarrow{PR} &= \langle -1, 3, -5 \rangle \times \langle 2, -3, -3 \rangle \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 3 & -5 \\ 2 & -3 & -3 \end{vmatrix} \\
&= \begin{vmatrix} 3 & -5 \\ -3 & -3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -1 & -5 \\ 2 & -3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -1 & 3 \\ 2 & -3 \end{vmatrix}\mathbf{k} \\
&= (-9-15)\mathbf{i} - (3-(-10))\mathbf{j} + (3-6)\mathbf{k} \\
&= -24\mathbf{i} - 13\mathbf{j} - 3\mathbf{k} \\
&= \langle -24, -13, -3 \rangle
\end{aligned}$$

For simplicity, we can use the opposite vector as our **n**: $\mathbf{n} = \langle 24, 13, 3 \rangle$.

Let's use $\mathbf{n} = \langle 24, 13, 3 \rangle$ as our normal and $P(0, 4, 2)$, say, as our point.

Standard form for an equation of the plane:

$$\boxed{24x + 13(y-4) + 3(z-2) = 0}$$

General form for an equation of the plane:

$$\boxed{24x + 13y + 3z - 58 = 0}$$

b) Find the area of triangle *PQR*, the triangle that has the three given points (*P*, *Q*, and *R*) as its vertices. You may refer to your work in part a). Round off your answer to the nearest tenth of a square meter.

$$\begin{aligned}
\text{Area} &= \frac{1}{2}\left\| \overrightarrow{PQ} \times \overrightarrow{PR} \right\| \\
&= \frac{1}{2}\sqrt{(-24)^2 + (-13)^2 + (-3)^2} \\
&= \frac{1}{2}\sqrt{754} \\
&\approx \boxed{13.7 \text{ square meters}}
\end{aligned}$$

10) Find the distance between the point $(5, 2, -4)$ and the plane $3x - y + 2z + 7 = 0$. Distance is measured in meters. Round off your answer to the nearest tenth of a meter. (8 points)

Method 1

$$\frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|3(5) - 1(2) + 2(-4) + 7|}{\sqrt{(3)^2 + (-1)^2 + (2)^2}}$$
$$= \frac{12}{\sqrt{14}}$$
$$\approx \boxed{3.2 \text{ meters}}$$

Method 2

Find a point on the given plane by, say, plugging in $x = 0$ and $z = 0$ into the equation of the plane and getting $(0, 7, 0)$.

Let the **p** vector connect this point, $(0, 7, 0)$, to the given point, $(5, 2, -4)$:

$$\begin{aligned} \mathbf{p} &= \langle 5 - 0, 2 - 7, -4 - 0 \rangle \\ &= \langle 5, -5, -4 \rangle \end{aligned}$$

A normal vector for the given plane is $\mathbf{n} = \langle 3, -1, 2 \rangle$.

The desired distance is:

$$\begin{aligned} |\text{comp}_{\mathbf{n}}\mathbf{p}| &= \frac{|\mathbf{p} \bullet \mathbf{n}|}{\|\mathbf{n}\|} \\ &= \frac{|\langle 5, -5, -4 \rangle \bullet \langle 3, -1, 2 \rangle|}{\|\langle 3, -1, 2 \rangle\|} \\ &= \frac{|15 + 5 - 8|}{\sqrt{(3)^2 + (-1)^2 + (2)^2}} \\ &= \frac{12}{\sqrt{14}} \\ &\approx \boxed{3.2 \text{ meters}} \end{aligned}$$

Note: The formula for the distance between a point and a line in the plane is inappropriate, because there is no guarantee that your line contains the point on the plane closest to the given point.

11) Matching. (12 points total)

Fill in each blank below with one of the following letters (A-F):

A. An Ellipsoid
B. A Hyperboloid of One Sheet
C. A Hyperboloid of Two Sheets
D. A Cone
E. An Elliptic Paraboloid
F. A Hyperbolic Paraboloid

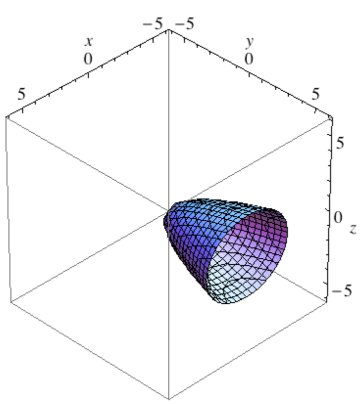
I. The graph of $3x^2 - 5y + 4z^2 = 0$ is __**E**__.
(Think: $x^2 - y + z^2 = 0$, or $y = x^2 + z^2$.)

II. The graph of $4z^2 - x^2 - 5y^2 = 2$ is __**C**__.
(Think: $z^2 - x^2 - y^2 = 1$, or $z^2 = x^2 + y^2 + 1$.)

III. The graph of $x^2 = y^2 + \frac{2}{5}z^2$ is __**D**__.
(Think: $x^2 = y^2 + z^2$.)

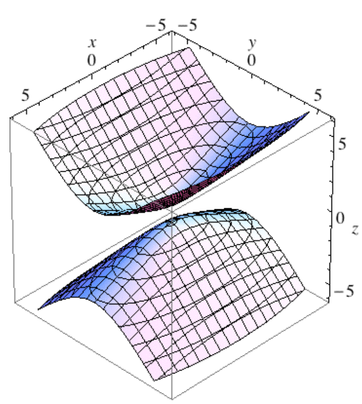
IV. The graph of $y = 2x^2 - 3z^2$ is __**F**__.
(Think: $y = x^2 - z^2$.)

Below are graphs of I, II, and III. IV will show up later! (Courtesy *Mathematica*.)



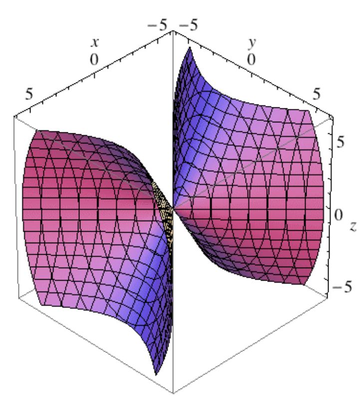
$3x^2 - 5y + 4z^2 = 0$

An Elliptic Paraboloid



$4z^2 - x^2 - 5y^2 = 2$

A Hyperboloid of Two Sheets



$x^2 = y^2 + \frac{2}{5}z^2$

A Cone

12) Consider the graph of $y = 2x^2 - 3z^2$. This was in Problem 11, part IV. Assume that k takes the place of real constants. (9 points total)

The graph is a hyperbolic paraboloid. For simplicity, consider: $y = x^2 - z^2$.
The conic traces of the graph in the planes $y = k$ are hyperbolas.
The conic traces in the planes $x = k$ and $z = k$ are parabolas.

a) The conic traces of the graph in the planes $x = k$ are … (Box in one:)

Ellipses Hyperbolas $\boxed{\textbf{Parabolas}}$

Let k be a real number. The trace of the graph of $y = x^2 - z^2$ in the plane $x = k$ is given by:

$$y = k^2 - z^2,\ \ x = k$$
$$y = -z^2 + k^2,\ \ x = k$$

We obtain parabolas.

b) The conic traces of the graph in the planes $y = k$ are … (Box in one:)

Ellipses $\boxed{\textbf{Hyperbolas}}$ Parabolas

Let k be a real number. The trace of the graph of $y = x^2 - z^2$ in the plane $y = k$ is given by:

$$k = x^2 - z^2,\ \ y = k$$
$$x^2 - z^2 = k,\ \ y = k$$

We obtain hyperbolas for $k \neq 0$ and a pair of intersecting lines for $k = 0$.

c) The conic traces of the graph in the planes $z = k$ are … (Box in one:)

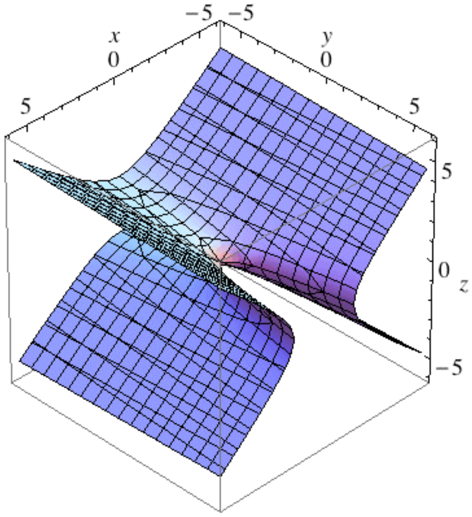
Ellipses Hyperbolas $\boxed{\textbf{Parabolas}}$

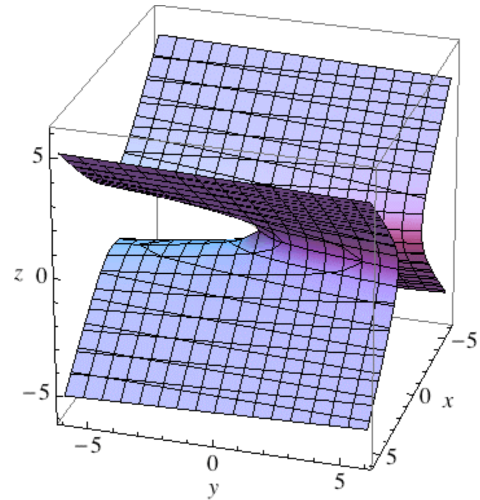
Let k be a real number. The trace of the graph of $y = x^2 - z^2$ in the plane $z = k$ is given by:

$$y = x^2 - k^2,\ \ z = k$$

We obtain parabolas.

Here are graphs of $y = x^2 - z^2$, a hyperbolic paraboloid:





13) Find an equation (in x, y, and z) of the surface obtained by revolving the graph of $16x^2 - 9z^2 = 1$ (in the xz-plane) about the z-axis. (3 points)

Since x is the "non-axis" variable in the equation above, and y is the "missing variable" …

We replace x^2 with $\left(x^2 + y^2\right)$. We don't "touch z," the "axis variable."

$$16\left(x^2 + y^2\right) - 9z^2 = 1$$

$$\boxed{16x^2 + 16y^2 - 9z^2 = 1}$$

We are taking a hyperbola (in blue below) that opens along the x-axis in the xz-plane (see left figure below), and we are using it to generate a hyperboloid of one sheet with the z-axis as its axis (see right figure below).