QUIZ 2 (CHAPTER 15, 16.1, 16.2) SOLUTIONS

MATH 252 – FALL 2007 – KUNIYUKI
SCORED OUT OF 125 POINTS ⇒ MULTIPLIED BY 0.84 ⇒ 105% POSSIBLE

1) Find the length of the curve parameterized by:

$$x = t^2, \quad y = \frac{\sqrt{5}}{2}t^2, \quad z = 2t, \quad 0 \le t \le 2.$$

Major Hint (which you may use without proof):
According to the Table of Integrals, if a is a positive real constant,

$$\int \sqrt{a^2 + u^2}\, du = \frac{u}{2}\sqrt{a^2 + u^2} + \frac{a^2}{2}\ln\left| u + \sqrt{a^2 + u^2} \right| + C$$

Leave your answer as a simplified exact answer; do not approximate it using a calculator. You do not have to apply log properties at the end. Distance is measured in meters. Show all work! (20 points)

$$\begin{aligned}
L &= \int_0^2 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt \\
&= \int_0^2 \sqrt{(2t)^2 + \left(\sqrt{5}t\right)^2 + (2)^2}\, dt \\
&= \int_0^2 \sqrt{4t^2 + 5t^2 + 4}\, dt \\
&= \int_0^2 \sqrt{4 + 9t^2}\, dt \\
&= \int_0^2 \sqrt{(2)^2 + (3t)^2}\, dt
\end{aligned}$$

Let $a = 2$, so that $a^2 = 4$.

Perform a classic u-substitution:

Let $u = 3t$. Then,

$$du = 3\, dt \quad \Rightarrow$$

$$dt = \frac{1}{3}\, du$$

Change the limits of integration:

$$t = 0 \quad \Rightarrow \quad u = 3(0) = 0$$

$$t = 2 \quad \Rightarrow \quad u = 3(2) = 6$$

$$L = \int_0^6 \sqrt{4+u^2} \cdot \frac{1}{3}\,du$$

$$= \frac{1}{3}\int_0^6 \sqrt{4+u^2}\,du$$

$$= \frac{1}{3}\left[\frac{u}{2}\sqrt{4+u^2} + \frac{\overbrace{4}^{=2}}{2}\ln\left|u+\sqrt{4+u^2}\right|\right]_0^6 \quad \text{(by the Table of Integrals hint)}$$

$$= \frac{1}{3}\left(\left[\frac{(6)}{2}\sqrt{4+(6)^2} + 2\ln\left|(6)+\sqrt{4+(6)^2}\right|\right] - \left[\frac{(0)}{2}\sqrt{4+(0)^2} + 2\ln\left|(0)+\sqrt{4+(0)^2}\right|\right]\right)$$

$$= \frac{1}{3}\left(\left[3\sqrt{40} + 2\ln\left|6+\sqrt{40}\right|\right] - \left[0 + 2\ln\left|\sqrt{4}\right|\right]\right)$$

$$= \frac{1}{3}\left[3\left(2\sqrt{10}\right) + 2\ln\left|6+2\sqrt{10}\right| - 2\ln|2|\right]$$

$$= \boxed{\begin{array}{l} \frac{1}{3}\left[6\sqrt{10} + 2\ln\left(6+2\sqrt{10}\right) - 2\ln 2\right] \text{ meters} \\ \text{or } 2\sqrt{10} + \ln\left(3+\sqrt{10}\right)^{2/3} \text{ meters} \quad \text{(See Note below.)} \end{array}}$$

<u>Note</u>:

$$\frac{1}{3}\left[6\sqrt{10} + 2\ln\left(6+2\sqrt{10}\right) - 2\ln 2\right] = \frac{1}{3}\left[6\sqrt{10} + \ln\left(6+2\sqrt{10}\right)^2 - \ln(2)^2\right]$$

$$= \frac{1}{3}\left[6\sqrt{10} + \ln\frac{\left(6+2\sqrt{10}\right)^2}{(2)^2}\right]$$

$$= \frac{1}{3}\left[6\sqrt{10} + \ln\left(\frac{6+2\sqrt{10}}{2}\right)^2\right]$$

$$= 2\sqrt{10} + \frac{1}{3}\ln\left(3+\sqrt{10}\right)^2$$

$$= 2\sqrt{10} + \ln\left(3+\sqrt{10}\right)^{2/3}$$

$$\approx 7.53685$$

2) A curve C is parameterized by the vector-valued function (VVF) given by $\mathbf{r}(t) = \langle e^{2t}, 3t+1, t^2 \rangle$. Find a tangent vector to C at the point $(e^{10}, 16, 25)$. (10 points)

Find the t-value corresponding to the given point.

Let's equate the y-components and solve:

$$\begin{aligned} 3t + 1 &= 16 \\ t &= 5 \end{aligned}$$

Check:

$$\begin{aligned} \mathbf{r}(5) &= \left\langle e^{2(5)}, 3(5)+1, (5)^2 \right\rangle \\ &= \left\langle e^{10}, 16, 25 \right\rangle \end{aligned}$$

Find $\mathbf{r}'(5)$.

$$\begin{aligned} \mathbf{r}(t) &= \left\langle e^{2t}, 3t+1, t^2 \right\rangle \Rightarrow \\ \mathbf{r}'(t) &= \left\langle 2e^{2t}, 3, 2t \right\rangle \Rightarrow \\ \mathbf{r}'(5) &= \left\langle 2e^{2(5)}, 3, 2(5) \right\rangle \\ &= \boxed{\left\langle 2e^{10}, 3, 10 \right\rangle} \quad (\text{or any non-}\mathbf{0}\text{ scalar multiple of this}) \end{aligned}$$

3) Complete the Product Rule for differentiating the dot product of two differentiable vector-valued functions (VVFs) $\mathbf{u}$ and $\mathbf{v}$:

$$D_t\left[\mathbf{u}(t) \bullet \mathbf{v}(t)\right] = \boxed{\mathbf{u}(t) \bullet \mathbf{v}'(t) + \mathbf{u}'(t) \bullet \mathbf{v}(t)}$$

(3 points)

4) The velocity of a moving particle is given by $\mathbf{v}(t) = \langle 7e^t,\ 4\cos t,\ 3t^2 - 2 \rangle$. Find the position vector-valued function (VVF rule) $\mathbf{r}(t)$ if $\mathbf{r}(0) = \langle 1, 4, -3 \rangle$. (10 points)

$$\begin{aligned} \mathbf{r}(t) &= \int \mathbf{v}(t)\,dt \quad (\text{one member}) \\ &= \int \langle 7e^t,\ 4\cos t,\ 3t^2 - 2 \rangle\,dt \\ &= \langle 7e^t,\ 4\sin t,\ t^3 - 2t \rangle + \mathbf{C} \end{aligned}$$

Solve for $\mathbf{C}$ by plugging in $t = 0$ and using the initial condition.

$$\begin{aligned} \mathbf{r}(0) &= \left\langle 7e^{(0)},\ 4\sin(0),\ (0)^3 - 2(0) \right\rangle + \mathbf{C} \\ \langle 1, 4, -3 \rangle &= \langle 7,\ 0,\ 0 \rangle + \mathbf{C} \\ \mathbf{C} &= \langle -6, 4, -3 \rangle \end{aligned}$$

Therefore,

$$\mathbf{r}(t) = \langle 7e^t,\ 4\sin t,\ t^3 \rangle + \langle -6, 4, -3 \rangle$$

$$\boxed{\mathbf{r}(t) = \langle 7e^t - 6,\ 4\sin t + 4,\ t^3 - 2t - 3 \rangle}$$

5) Find the unit tangent VVF (rule) $\mathbf{T}(t)$ and the principal unit normal VVF (rule) $\mathbf{N}(t)$ for the curve C determined by $\mathbf{r}(t) = \langle -6t, 2t^3 \rangle$, where $t > 0$. Show all work and simplify completely, as we have done in class. Do **not** use the fact that $\mathbf{T}(t) \perp \mathbf{N}(t)$, and do **not** eliminate the parameter. Messy and/or undisciplined work may not be graded! (27 points)

$$\mathbf{r}(t) = \langle -6t, 2t^3 \rangle$$

$$\mathbf{r}'(t) = \langle -6, 6t^2 \rangle, \text{ or } 6\langle -1, t^2 \rangle$$

$$\begin{aligned} \|\mathbf{r}'(t)\| &= \|\langle -6, 6t^2 \rangle\| \\ &= |6|\,\|\langle -1, t^2 \rangle\| \\ &= 6\sqrt{(-1)^2 + (t^2)^2} \\ &= 6\sqrt{1 + t^4} \end{aligned}$$

$$\mathbf{T}(t)=\frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$$

$$=\frac{\langle -6, 6t^2\rangle}{6\sqrt{1+t^4}}$$

$$=\frac{\cancel{6}\langle -1, t^2\rangle}{\cancel{6}\sqrt{1+t^4}}$$

$$=\frac{\langle -1, t^2\rangle}{\sqrt{1+t^4}}$$

$$\mathbf{T}(t)=\boxed{\left\langle -\frac{1}{\sqrt{1+t^4}},\ \frac{t^2}{\sqrt{1+t^4}}\right\rangle \text{ or } \left\langle -(1+t^4)^{-1/2},\ \frac{t^2}{\sqrt{1+t^4}}\right\rangle}$$

$$\mathbf{T}'(t)=\left\langle \frac{1}{2}(1+t^4)^{-3/2}(4t^3),\ \frac{\left(\sqrt{1+t^4}\right)\left[D_t\left(t^2\right)\right]-\left(t^2\right)\left[D_t\left(1+t^4\right)^{1/2}\right]}{\left(\sqrt{1+t^4}\right)^2}\right\rangle$$

(for the second component, we used the Quotient Rule for Diff.)

$$=\left\langle \frac{2t^3}{(1+t^4)^{3/2}},\ \frac{\left(\sqrt{1+t^4}\right)(2t)-\left(t^2\right)\left[\frac{1}{2}(1+t^4)^{-1/2}(4t^3)\right]}{1+t^4}\right\rangle$$

$$=\left\langle \frac{2t^3}{(1+t^4)^{3/2}},\ \frac{\left(2t\sqrt{1+t^4}-\frac{2t^5}{\sqrt{1+t^4}}\right)}{(1+t^4)}\cdot\frac{\sqrt{1+t^4}}{\sqrt{1+t^4}}\right\rangle$$

$$=\left\langle \frac{2t^3}{(1+t^4)^{3/2}},\ \frac{2t(1+t^4)-2t^5}{(1+t^4)^{3/2}}\right\rangle$$

$$=\left\langle \frac{2t^3}{(1+t^4)^{3/2}},\ \frac{2t+\cancel{2t^5}-\cancel{2t^5}}{(1+t^4)^{3/2}}\right\rangle$$

$$=\left\langle \frac{2t^3}{(1+t^4)^{3/2}},\ \frac{2t}{(1+t^4)^{3/2}}\right\rangle$$

$$=\frac{2t}{(1+t^4)^{3/2}}\left\langle t^2, 1\right\rangle$$

$$\left\| \mathbf{T}'(t) \right\| = \left\| \frac{2t}{\left(1+t^4\right)^{3/2}} \left\langle t^2, 1 \right\rangle \right\|$$

$$= \left| \frac{2t}{\left(1+t^4\right)^{3/2}} \right| \left\| \left\langle t^2, 1 \right\rangle \right\|$$

$$= \frac{2t}{\left(1+t^4\right)^{3/2}} \sqrt{\left(t^2\right)^2 + \left(1\right)^2} \quad \left(\text{We assume } t > 0.\right)$$

$$= \frac{2t}{\left(1+t^4\right)^{3/2}} \sqrt{t^4 + 1}$$

$$= \frac{2t}{\left(1+t^4\right)^{3/2}} \cdot \left(1+t^4\right)^{1/2}$$

$$= \frac{2t}{1+t^4}$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\left\| \mathbf{T}'(t) \right\|}$$

$$= \frac{\dfrac{2t}{\left(1+t^4\right)^{3/2}} \left\langle t^2, 1 \right\rangle}{\dfrac{2t}{1+t^4}}$$

$$= \frac{\cancel{2t}}{\left(1+t^4\right)^{3/2}} \cdot \frac{1+t^4}{\cancel{2t}} \left\langle t^2, 1 \right\rangle$$

$$= \frac{1}{\sqrt{1+t^4}} \left\langle t^2, 1 \right\rangle$$

$$\boxed{\mathbf{N}(t) = \frac{1}{\sqrt{1+t^4}} \left\langle t^2, 1 \right\rangle \text{ or } \left\langle \frac{t^2}{\sqrt{1+t^4}}, \frac{1}{\sqrt{1+t^4}} \right\rangle}$$

Observe: For $t > 0$, $\mathbf{T}(t) \bullet \mathbf{N}(t) = 0$, which reflects the fact that $\mathbf{T}(t) \perp \mathbf{N}(t)$.

Note: If we had eliminated the parameter, we would have obtained $y = -\frac{x^3}{108}$, which can be analyzed directly.

6) Assume that **r** is a position VVF of t in 3-space that is twice differentiable everywhere (i.e., second derivatives exist for all real t). Write a curvature formula we discussed for $\kappa(t)$ that involves a cross product. (4 points)

$$\boxed{\kappa(t) = \frac{\|\mathbf{v}(t)\times\mathbf{a}(t)\|}{\|\mathbf{v}(t)\|^3} \quad \text{or} \quad \frac{\|\mathbf{r}'(t)\times\mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}}$$

7) A helical curve C is determined by $\mathbf{r}(t) = \langle 2\cos t,\ 2\sin t,\ 3t\rangle$. The curvature at every point on the curve is given by a constant, κ. Find κ. Use your formula from Problem 6), and simplify your answer completely. Show all work! (16 points)

Let $\mathbf{r}(t) = \langle 2\cos t,\ 2\sin t,\ 3t\rangle$.

Then, $\mathbf{r}'(t)$ or $\mathbf{v}(t) = \langle -2\sin t,\ 2\cos t,\ 3\rangle$, and

$\mathbf{r}''(t)$ or $\mathbf{a}(t) = \langle -2\cos t,\ -2\sin t,\ 0\rangle$.

$$\begin{aligned}
\mathbf{v}(t)\times\mathbf{a}(t) &= \langle -2\sin t,\ 2\cos t,\ 3\rangle \times \langle -2\cos t,\ -2\sin t,\ 0\rangle \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2\sin t & 2\cos t & 3 \\ -2\cos t & -2\sin t & 0 \end{vmatrix} \\
&= \begin{vmatrix} 2\cos t & 3 \\ -2\sin t & 0 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -2\sin t & 3 \\ -2\cos t & 0 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -2\sin t & 2\cos t \\ -2\cos t & -2\sin t \end{vmatrix}\mathbf{k} \\
&= (6\sin t)\mathbf{i} - (6\cos t)\mathbf{j} + \underbrace{\left(4\sin^2 t + 4\cos^2 t\right)}_{\substack{=4(\sin^2 t+\cos^2 t)\\ =4(1)\\ =4}}\mathbf{k} \\
&= (6\sin t)\mathbf{i} - (6\cos t)\mathbf{j} + 4\mathbf{k} \\
&= \langle 6\sin t,\ -6\cos t,\ 4\rangle \text{ or } 2\langle 3\sin t,\ -3\cos t,\ 2\rangle
\end{aligned}$$

$$
\begin{aligned}
\left\|\mathbf{v}(t)\times\mathbf{a}(t)\right\| &= \left\|\langle 6\sin t,\ -6\cos t,\ 4\rangle\right\| \\
&= \sqrt{(6\sin t)^2+(-6\cos t)^2+(4)^2} \\
&= \sqrt{36\sin^2 t+36\cos^2 t+16} \\
&= \sqrt{36\Big(\underbrace{\sin^2 t+\cos^2 t}_{=1}\Big)+16} \\
&= \sqrt{36+16} \\
&= \sqrt{52} \\
&= 2\sqrt{13}
\end{aligned}
$$

or

$$
\begin{aligned}
&= 2\left\|\langle 3\sin t,\ -3\cos t,\ 2\rangle\right\| \\
&= 2\sqrt{(3\sin t)^2+(-3\cos t)^2+(2)^2} \\
&= 2\sqrt{9\sin^2 t+9\cos^2 t+4} \\
&= 2\sqrt{9\Big(\underbrace{\sin^2 t+\cos^2 t}_{=1}\Big)+4} \\
&= 2\sqrt{9+4} \\
&= 2\sqrt{13}
\end{aligned}
$$

$$
\begin{aligned}
\left\|\mathbf{v}(t)\right\|^3 &= \left\|\langle -2\sin t,\ 2\cos t,\ 3\rangle\right\|^3 \\
&= \left(\sqrt{(-2\sin t)^2+(2\cos t)^2+(3)^2}\right)^3 \\
&= \left(\sqrt{4\sin^2 t+4\cos^2 t+9}\right)^3 \\
&= \left(\sqrt{4\Big(\underbrace{\sin^2 t+\cos^2 t}_{=1}\Big)+9}\right)^3 \\
&= \left(\sqrt{4+9}\right)^3 \\
&= \left(\sqrt{13}\right)^3 \\
&= 13\sqrt{13}
\end{aligned}
$$

$$
\kappa(t)=\frac{\left\|\mathbf{v}(t)\times\mathbf{a}(t)\right\|}{\left\|\mathbf{v}(t)\right\|^3}=\frac{2\sqrt{13}}{13\sqrt{13}}=\boxed{\frac{2}{13}}
$$

8) Sketch the level curves of $f(x,y)=(x-1)^2+y^2$ for $k = 1, 4, 9$ on the grid below. Label the curves with their corresponding k-values. Be reasonably accurate. (8 points)

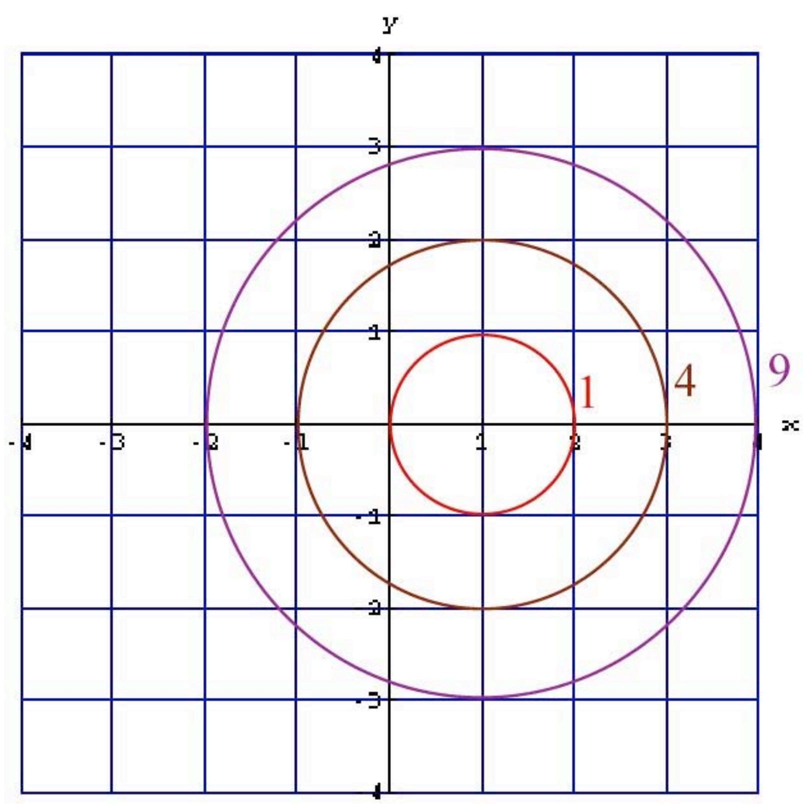
Let $f(x,y)=k$.

For any real k:

$$k=(x-1)^2+y^2$$

The graph of this is a circle of radius $\sqrt{k}$ centered at the point $(1,0)$ in the xy-plane.

$k=1$: The circle has radius 1.
$k=4$: The circle has radius 2.
$k=9$: The circle has radius 3.



9) What is the graph of $z = (x-1)^2 + y^2$ in *xyz*-space? Problem 8) may help.
Box in one: (3 points)

A Cone $\boxed{\text{A Paraboloid}}$ A Sphere

The original graph of f is a paraboloid with equation $z = (x-1)^2 + y$, so it makes sense that the level curves are circles. The paraboloid opens upward, and its vertex is at $(1,0,0)$.

10) Matching. (9 points total)

Fill in each blank with the best choice in the list below to indicate the level surface of f for the given value of *k*.

A. A Sphere or Ellipsoid
B. A Hyperboloid of One Sheet
C. A Hyperboloid of Two Sheets
D. A Cone
E. A Circular or Elliptic Paraboloid
F. A Hyperbolic Paraboloid
G. A Right Circular or Elliptic Cylinder
H. A Plane
I. A Line (a "degenerate" surface)
J. A Point (a "degenerate" surface)
K. NONE (no surface)

a) The level surface of $f(x,y,z) = 2x - 4y + 5z,\ k = 10$ is __**H**__.

Analyze: $10 = 2x - 4y + 5z$. This is a nondegenerate linear equation in *x*, *y*, and *z*, so its graph is a plane in *xyz*-space.

b) The level surface of $f(x,y,z) = x^2 + y^2 - z^2,\ k = 4$ is __**B**__.

Analyze: $4 = x^2 + y^2 - z^2$. Its graph is a hyperboloid of one sheet in *xyz*-space. Its axis is the *z*-axis.

c) The level surface of $f(x,y,z) = x^2 + y^2 - z^2,\ k = -4$ is __**C**__.

Analyze: $-4 = x^2 + y^2 - z^2$, which is equivalent to: $-x^2 - y^2 + z^2 = 4$.
The graph is a hyperboloid of two sheets in *xyz*-space. Its axis is the *z*-axis.

11) Show that $\lim_{(x,y)\to(0,0)} \frac{2x^3 + y^3}{5x^3 - 2y^3}$ does not exist. (10 points)

Let $(x,y)\to(0,0)$ along the y-axis ($x = 0$):

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{2x^3 + y^3}{5x^3 - 2y^3} &= \lim_{(x,y)\to(0,0)} \frac{2(0)^3 + y^3}{5(0)^3 - 2y^3} \\ &= \lim_{y\to 0} \frac{y^3}{-2y^3} \\ &= -\frac{1}{2}\end{aligned}$$

Let $(x,y)\to(0,0)$ along the x-axis ($y = 0$):

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{2x^3 + y^3}{5x^3 - 2y^3} &= \lim_{(x,y)\to(0,0)} \frac{2x^3 + (0)^3}{5x^3 - 2(0)^3} \\ &= \lim_{x\to 0} \frac{2x^3}{5x^3} \\ &= \frac{2}{5}\end{aligned}$$

We have found two paths approaching $(0, 0)$ that yield different limit values for $\frac{2x^3 + y^3}{5x^3 - 2y^3}$, so the indicated limit does not exist by the Two-Path Rule.

12) Use polar coordinates to find $\lim_{(x,y)\to(0,0)} \frac{\sin(x^2 + y^2)}{x^2 + y^2}$. (5 points)

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{\sin(x^2 + y^2)}{x^2 + y^2} &= \lim_{r\to 0} \frac{\sin(r^2)}{r^2} && \left(\frac{0}{0} \text{ limit form}\right) \\ &= \lim_{r\to 0} \frac{D_r\left[\sin(r^2)\right]}{D_r\left[r^2\right]} && (\text{by L'Hôpital's Rule}) \\ &= \lim_{r\to 0} \frac{{}^{1}\cancel{2r}\cos(r^2)}{{}_{1}\cancel{2r}} \\ &= \cos(0) \\ &= \boxed{1}\end{aligned}$$