QUIZ 2 (CHAPTER 15, 16.1, 16.2) SOLUTIONS

MATH 252 – FALL 2008 – KUNIYUKI
SCORED OUT OF 125 POINTS ⇒ MULTIPLIED BY 0.84 ⇒ 105% POSSIBLE

Clearly mark vectors, as we have done in class. I will use boldface, but you don't! When describing vectors or vector-valued functions, you may use either $\langle\ \rangle$ or "$\mathbf{i}-\mathbf{j}-\mathbf{k}$" notation. Assume we are in our usual 2- and 3-dimensional Cartesian coordinate systems. Give exact answers, unless otherwise specified.

1) Find the length of the curve parameterized by:

$$x = t\sin t + \cos t, \quad y = t\cos t - \sin t, \quad z = \frac{3}{2}t^2, \quad 0 \le t \le 4.$$

Leave your answer as a simplified exact answer; do not approximate it using a calculator. Distance is measured in meters. Show all work! (20 points)

$$L = \int_0^4 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

$$= \int_0^4 \sqrt{\left(D_t\left[t\sin t + \cos t\right]\right)^2 + \left(D_t\left[t\cos t - \sin t\right]\right)^2 + \left(D_t\left[\frac{3}{2}t^2\right]\right)^2}\, dt$$

Use the Product Rule for $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

$$= \int_0^4 \sqrt{\left(\cancel{\sin t} + t\cos t - \cancel{\sin t}\right)^2 + \left(\cancel{\cos t} - t\sin t - \cancel{\cos t}\right)^2 + \left(3t\right)^2}\, dt$$

$$= \int_0^4 \sqrt{\left(t\cos t\right)^2 + \left(-t\sin t\right)^2 + 9t^2}\, dt$$

$$= \int_0^4 \sqrt{t^2\cos^2 t + t^2\sin^2 t + 9t^2}\, dt$$

$$= \int_0^4 \sqrt{t^2\left(\cos^2 t + \sin^2 t\right) + 9t^2}\, dt$$

$$= \int_0^4 \sqrt{t^2 + 9t^2}\, dt$$

$$= \int_0^4 \sqrt{10t^2}\, dt$$

$$= \sqrt{10}\int_0^4 t\, dt$$

Observe: $\sqrt{t^2} = \left|t\right| = t$, since $t \ge 0$ on the interval $\left[0, 4\right]$.
Now, apply the Fundamental Theorem of Calculus.

$$= \sqrt{10}\left[\frac{t^2}{2}\right]_0^4$$

$$= \sqrt{10}\left(\left[\frac{(4)^2}{2}\right] - \left[\frac{(0)^2}{2}\right]\right)$$

$$= \sqrt{10}\,(8)$$

$$= \boxed{8\sqrt{10}\ \text{meters}}$$

This is about 25.2982 meters.

2) Find parametric equations for the tangent line to C at the point $(25, 0, 478)$, where C is parameterized by: $x = t^3 - 2,\quad y = \sin(\pi t),\quad z = 2t^5 - t^2 + 1$. (12 points)

$$\mathbf{r}(t) = \langle t^3 - 2,\ \sin(\pi t),\ 2t^5 - t^2 + 1\rangle .$$

Find the t-value corresponding to the given point.

It helps that x is a one-to-one function of t. Solve $t^3 - 2 = 25$.

$$\begin{aligned} t^3 - 2 &= 25 \\ t^3 &= 27 \\ t &= 3 \end{aligned}$$

Check:

$$\begin{aligned} \mathbf{r}(3) &= \left\langle (3)^3 - 2,\ \sin(3\pi),\ 2(3)^5 - (3)^2 + 1\right\rangle \\ &= \langle 25,\ 0,\ 478\rangle \end{aligned}$$

Find a direction vector for the desired tangent line.

$$\begin{aligned} \mathbf{r}(t) &= \langle t^3 - 2,\ \sin(\pi t),\ 2t^5 - t^2 + 1\rangle \\ \mathbf{r}'(t) &= \langle 3t^2,\ [\cos(\pi t)][\pi],\ 10t^4 - 2t\rangle \\ &= \langle 3t^2,\ \pi\cos(\pi t),\ 10t^4 - 2t\rangle \\ \mathbf{r}'(3) &= \left\langle 3(3)^2,\ \pi\cos(3\pi),\ 10(3)^4 - 2(3)\right\rangle \\ &= \langle 27,\ \pi(-1),\ 804\rangle \\ &= \langle 27,\ -\pi,\ 804\rangle \quad \text{(You can use any non-}\mathbf{0}\text{ scalar multiple of this.)} \end{aligned}$$

Write parametric equations for the desired tangent line.

$$\begin{aligned} x &= 25 + 27u \\ y &= 0 - \pi u \qquad (u \text{ in } \mathbf{R}) \\ z &= 478 + 804u \end{aligned}$$

$$\boxed{\begin{aligned} x &= 25 + 27u \\ y &= \quad - \pi u \qquad (u \text{ in } \mathbf{R}) \\ z &= 478 + 804u \end{aligned}}$$

3) A curve C in 3-space is smoothly parameterized by the position vector-valued function (VVF) rule $\mathbf{r}(t)$. The position vector $\mathbf{r}(t)$ and the tangent vector $\mathbf{r}'(t)$ are orthogonal for all real t. Simplify $D_t\left[\mathbf{r}(t) \bullet \mathbf{r}(t)\right]$ for all real t. Use a differentiation rule discussed in class. (5 points)

$$D_t\left[\mathbf{r}(t) \bullet \mathbf{r}(t)\right] = \underbrace{\mathbf{r}'(t) \bullet \mathbf{r}(t)}_{\substack{=0 \\ \text{by orthogonality}}} + \underbrace{\mathbf{r}(t) \bullet \mathbf{r}'(t)}_{\substack{=0 \\ \text{by orthogonality}}}$$

(by a Product Rule for VVFs: (iii) on p.756)

$$= \boxed{0}$$

4) The acceleration of a moving particle is given by $\mathbf{a}(t) = (3\sin t)\mathbf{i} + (5\cos t)\mathbf{j}$. Find the position vector-valued function (VVF rule) $\mathbf{r}(t)$ if $\mathbf{r}(0) = 2\mathbf{i} - 6\mathbf{j}$ and $\mathbf{v}(0) = \mathbf{i} + 3\mathbf{j}$. (15 points)

$$\begin{aligned} \mathbf{v}(t) &= \int \mathbf{a}(t)\,dt \quad (\text{one member}) \\ &= \int \langle 3\sin t, 5\cos t \rangle\,dt \\ &= \langle -3\cos t,\ 5\sin t \rangle + \mathbf{C} \end{aligned}$$

Solve for **C** by plugging in $t = 0$ and using the initial condition $\mathbf{v}(0) = \mathbf{i} + 3\mathbf{j}$, or $\langle 1, 3 \rangle$.

$$\begin{aligned} \mathbf{v}(0) &= \langle -3\cos(0),\ 5\sin(0) \rangle + \mathbf{C} \\ \langle 1, 3 \rangle &= \langle -3(1),\ 5(0) \rangle + \mathbf{C} \\ \langle 1, 3 \rangle &= \langle -3,\ 0 \rangle + \mathbf{C} \\ \mathbf{C} &= \langle 4, 3 \rangle \end{aligned}$$

Therefore,

$$\begin{aligned} \mathbf{v}(t) &= \langle -3\cos t,\ 5\sin t \rangle + \langle 4, 3 \rangle \\ \mathbf{v}(t) &= \langle -3\cos t + 4,\ 5\sin t + 3 \rangle \end{aligned}$$

Now,

$$\begin{aligned} \mathbf{r}(t) &= \int \mathbf{v}(t)\,dt \quad (\text{one member}) \\ &= \int \langle -3\cos t + 4,\ 5\sin t + 3 \rangle\,dt \\ &= \langle -3\sin t + 4t,\ -5\cos t + 3t \rangle + \mathbf{D} \end{aligned}$$

Solve for **D** by plugging in $t = 0$ and using the initial condition $\mathbf{r}(0) = 2\mathbf{i} - 6\mathbf{j}$, or $\langle 2, -6 \rangle$.

$$\begin{aligned} \mathbf{r}(0) &= \langle -3\sin(0) + 4(0),\ -5\cos(0) + 3(0) \rangle + \mathbf{D} \\ \langle 2, -6 \rangle &= \langle 0,\ -5 \rangle + \mathbf{D} \\ \mathbf{D} &= \langle 2, -1 \rangle \end{aligned}$$

Therefore,

$$\mathbf{r}(t) = \langle -3\sin t + 4t,\ -5\cos t + 3t \rangle + \langle 2, -1 \rangle$$

$$\boxed{\begin{aligned} \mathbf{r}(t) &= \langle -3\sin t + 4t + 2,\ -5\cos t + 3t - 1 \rangle, \text{ or} \\ &\quad (-3\sin t + 4t + 2)\mathbf{i} + (-5\cos t + 3t - 1)\mathbf{j} \end{aligned}}$$

5) The curve C is determined by $\mathbf{r}(t) = \left\langle \frac{t^3}{3}, \frac{t^2}{2} \right\rangle$, where $t > 0$. Show all work, simplify radicals, and simplify completely, as we have done in class. Do **not** eliminate the parameter. Messy and/or undisciplined work may not be graded!

Note: When differentiating, avoid using the Product Rule as an alternative to the Quotient Rule, unless you simplify your result to the most compact form! (18 points total)

a) Find the unit tangent VVF (rule) $\mathbf{T}(t)$ for C.

$$\mathbf{r}(t) = \left\langle \frac{t^3}{3}, \frac{t^2}{2} \right\rangle$$

$$\mathbf{r}'(t) = \langle t^2, t \rangle, \text{ or } t\langle t, 1 \rangle$$

$$\begin{aligned} \|\mathbf{r}'(t)\| &= \|\langle t^2, t \rangle\|, \text{ or } \|t\langle t, 1 \rangle\| \\ &= |t| \, \|\langle t, 1 \rangle\| \\ &= t \, \|\langle t, 1 \rangle\| \quad \text{(because we can assume } t > 0\text{)} \\ &= t\sqrt{(t)^2 + (1)^2} \\ &= t\sqrt{t^2 + 1} \end{aligned}$$

$$\begin{aligned} \mathbf{T}(t) &= \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} \\ &= \frac{t\langle t, 1 \rangle}{t\sqrt{t^2+1}} \\ &= \frac{\langle t, 1 \rangle}{\sqrt{t^2+1}} \end{aligned}$$

$$\boxed{\mathbf{T}(t) = \frac{\langle t, 1 \rangle}{\sqrt{t^2+1}}, \text{ or } \mathbf{T}(t) = \left\langle \frac{t}{\sqrt{t^2+1}}, \frac{1}{\sqrt{t^2+1}} \right\rangle}$$

Note: If we had eliminated the parameter, we would have obtained $y = \frac{\sqrt[3]{9x^2}}{2}$, which can be analyzed directly.

b) Find the VVF (rule) $\mathbf{T}'(t)$ for C.

Remember, $\mathbf{T}(t) = \left\langle \frac{t}{\sqrt{t^2+1}}, \frac{1}{\sqrt{t^2+1}} \right\rangle$ or $\left\langle \frac{t}{\sqrt{t^2+1}}, \left(t^2+1\right)^{-\frac{1}{2}} \right\rangle$

$$\mathbf{T}'(t) = \left\langle \frac{\left[\sqrt{t^2+1}\right]\left[D_t(t)\right] - [t]\left[D_t\left(t^2+1\right)^{1/2}\right]}{\left(\sqrt{t^2+1}\right)^2}, \quad -\frac{1}{2}\left(t^2+1\right)^{-3/2}(2t) \right\rangle$$

(for the first component, we used the Quotient Rule for Diff.)

$$= \left\langle \frac{\left[\sqrt{t^2+1}\right][1] - [t]\left[\frac{1}{2}\left(t^2+1\right)^{-1/2}(2t)\right]}{\left(\sqrt{t^2+1}\right)^2}, \quad -\left(t^2+1\right)^{-3/2}(t) \right\rangle$$

$$= \left\langle \frac{\sqrt{t^2+1} - \frac{t^2}{\sqrt{t^2+1}}}{t^2+1}, \quad -\frac{t}{\left(t^2+1\right)^{3/2}} \right\rangle$$

$$= \left\langle \frac{\left(\sqrt{t^2+1} - \frac{t^2}{\sqrt{t^2+1}}\right)}{\left(t^2+1\right)} \cdot \frac{\sqrt{t^2+1}}{\sqrt{t^2+1}}, \quad -\frac{t}{\left(t^2+1\right)^{3/2}} \right\rangle$$

$$= \left\langle \frac{\left(t^2+1\right) - t^2}{\left(t^2+1\right)^{3/2}}, \quad -\frac{t}{\left(t^2+1\right)^{3/2}} \right\rangle$$

$$= \boxed{\left\langle \frac{1}{\left(t^2+1\right)^{3/2}}, -\frac{t}{\left(t^2+1\right)^{3/2}} \right\rangle, \text{ or } \frac{1}{\left(t^2+1\right)^{3/2}}\langle 1, -t\rangle}$$

You didn't have to do this, but here's how we can find $\mathbf{N}(t)$:

$$\begin{aligned}
\left\|\mathbf{T}'(t)\right\| &= \left\| \frac{1}{\left(t^2+1\right)^{3/2}} \langle 1, -t \rangle \right\| \\
&= \left| \frac{1}{\left(t^2+1\right)^{3/2}} \right| \left\| \langle 1, -t \rangle \right\| \\
&= \frac{1}{\left(t^2+1\right)^{3/2}} \sqrt{(1)^2 + (-t)^2} \\
&= \frac{1}{\left(t^2+1\right)^{3/2}} \sqrt{1+t^2} \\
&= \frac{1}{\left(t^2+1\right)^{3/2}} \cdot \left(t^2+1\right)^{1/2} \\
&= \frac{1}{t^2+1}
\end{aligned}$$

$$\begin{aligned}
\mathbf{N}(t) &= \frac{\mathbf{T}'(t)}{\left\|\mathbf{T}'(t)\right\|} \\
&= \frac{\dfrac{1}{\left(t^2+1\right)^{3/2}} \langle 1, -t \rangle}{\dfrac{1}{t^2+1}} \\
&= \frac{t^2+1}{1} \cdot \frac{1}{\left(t^2+1\right)^{3/2}} \langle 1, -t \rangle \\
&= \frac{1}{\left(t^2+1\right)^{1/2}} \langle 1, -t \rangle \\
&= \boxed{\left\langle \frac{1}{\sqrt{t^2+1}}, -\frac{t}{\sqrt{t^2+1}} \right\rangle}
\end{aligned}$$

Observe: For all $t > 0$, $\mathbf{N}(t) \bullet \mathbf{T}(t) = 0$, which reflects the fact that $\mathbf{N}(t) \perp \mathbf{T}(t)$.

6) Assume that **r** is a position VVF of *t* in 3-space that is twice differentiable everywhere (i.e., second derivatives exist for all real *t*). Write a curvature formula we discussed for $\kappa(t)$ that involves a cross product. (4 points)

$$\kappa(t) = \frac{\left\| \mathbf{v}(t) \times \mathbf{a}(t) \right\|}{\left\| \mathbf{v}(t) \right\|^3} \quad \text{or} \quad \frac{\left\| \mathbf{r}'(t) \times \mathbf{r}''(t) \right\|}{\left\| \mathbf{r}'(t) \right\|^3}$$

7) A twisted cubic curve *C* is determined by $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$, where $t > 0$.

(19 points total)

a) Find a general curvature formula, $\kappa(t)$, for every point on *C*. Use your formula from Problem 6), and simplify your answer completely. Show all work! (15 points)

Let $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$.

Then, $\mathbf{r}'(t)$ or $\mathbf{v}(t) = \langle 1, 2t, 3t^2 \rangle$, and

$\mathbf{r}''(t)$ or $\mathbf{a}(t) = \langle 0, 2, 6t \rangle$.

$$\begin{aligned}
\mathbf{v}(t) \times \mathbf{a}(t) &= \langle 1, 2t, 3t^2 \rangle \times \langle 0, 2, 6t \rangle \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix} \\
&= \begin{vmatrix} 2t & 3t^2 \\ 2 & 6t \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 3t^2 \\ 0 & 6t \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 2t \\ 0 & 2 \end{vmatrix} \mathbf{k} \\
&= (12t^2 - 6t^2)\mathbf{i} - (6t - 0)\mathbf{j} + (2 - 0)\mathbf{k} \\
&= 6t^2\,\mathbf{i} - 6t\,\mathbf{j} + 2\mathbf{k} \\
&= \langle 6t^2, -6t, 2 \rangle \quad \text{or} \quad 2\langle 3t^2, -3t, 1 \rangle
\end{aligned}$$

$$\begin{aligned}
\left\| \mathbf{v}(t) \times \mathbf{a}(t) \right\| &= \left\| \langle 6t^2, -6t, 2 \rangle \right\| \\
&= \sqrt{(6t^2)^2 + (-6t)^2 + (2)^2} \\
&= \sqrt{36t^4 + 36t^2 + 4} \\
&= \sqrt{4(9t^4 + 9t^2 + 1)} \\
&= 2\sqrt{9t^4 + 9t^2 + 1}
\end{aligned}$$

or

$$\begin{aligned}
&= 2\left\| \langle 3t^2, -3t, 1 \rangle \right\| \\
&= 2\sqrt{(3t^2)^2 + (-3t)^2 + (1)^2} \\
&= 2\sqrt{9t^4 + 9t^2 + 1}
\end{aligned}$$

$$\|\mathbf{v}(t)\|^3 = \|\langle 1, 2t, 3t^2 \rangle\|^3$$

$$= \left(\sqrt{(1)^2 + (2t)^2 + (3t^2)^2}\right)^3$$

$$= \left(\sqrt{1 + 4t^2 + 9t^4}\right)^3$$

$$= (9t^4 + 4t^2 + 1)^{3/2}$$

$$\kappa(t) = \frac{\|\mathbf{v}(t) \times \mathbf{a}(t)\|}{\|\mathbf{v}(t)\|^3} = \boxed{\frac{2\sqrt{9t^4 + 9t^2 + 1}}{(9t^4 + 4t^2 + 1)^{3/2}}}$$

b) Use your formula in part a) to find the curvature of the twisted cubic curve C at the point $(2, 4, 8)$. Approximate your answer to four decimal places. (4 points)

The point $(2, 4, 8)$ corresponds to $t = 2$.

$$\kappa(2) = \left.\frac{2\sqrt{9t^4 + 9t^2 + 1}}{(9t^4 + 4t^2 + 1)^{3/2}}\right|_{t=2}$$

$$= \frac{2\sqrt{9(2)^4 + 9(2)^2 + 1}}{\left(9(2)^4 + 4(2)^2 + 1\right)^{3/2}}$$

$$\approx \boxed{0.0132}$$

8) Sketch the level curves of $f(x, y) = y - x^2$ for $k = -3, 0, 3$ on the grid below. Label the curves with their corresponding k-values. Be reasonably accurate. (8 points)

Let $f(x, y) = k$. For any real k:

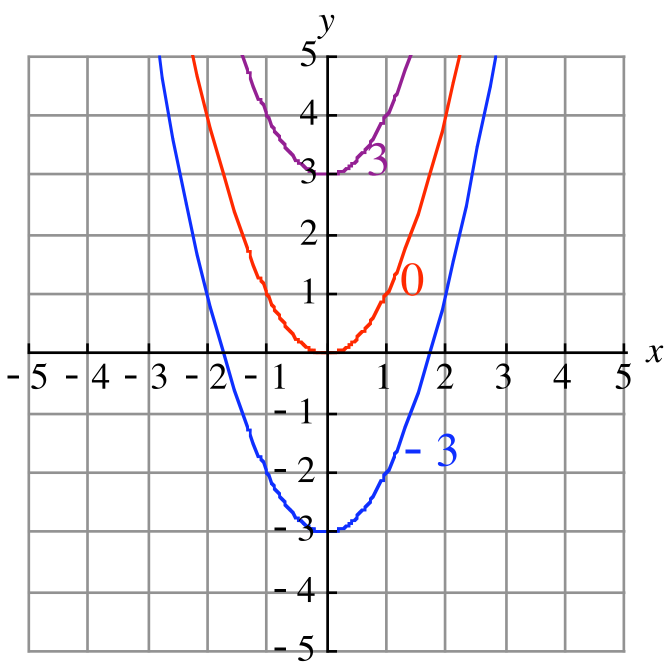
$$k = y - x^2$$
$$x^2 + k = y$$
$$y = x^2 + k$$

The graph of this is a parabola that opens upward and that has the point $(0, k)$ as its vertex and its y-intercept.

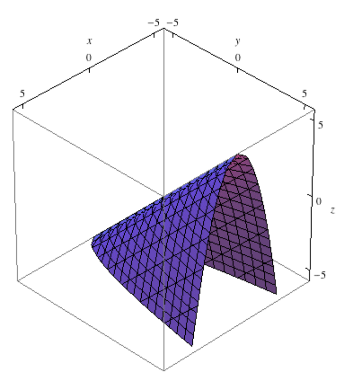
$k = -3$: The parabola $y = x^2 - 3$ has vertex $(0, -3)$.

$k = 0$: The parabola $y = x^2$ has vertex $(0, 0)$.

$k = 3$: The parabola $y = x^2 + 3$ has vertex $(0, 3)$.



Here is the corresponding surface, the graph of $z = y - x^2$:



9) Matching. (9 points total)

Fill in each blank with the best choice (A-K) in the list below to indicate the level surface of f for the given value of *k*.

A. A Sphere or Ellipsoid
B. A Hyperboloid of One Sheet
C. A Hyperboloid of Two Sheets
D. A Cone
E. A Circular or Elliptic Paraboloid
F. A Hyperbolic Paraboloid
G. A Right Circular or Elliptic Cylinder
H. A Plane
I. A Line (a "degenerate" surface)
J. A Point (a "degenerate" surface)
K. NONE (no surface)

a) The level surface of $f\left(x,y,z\right)=x^2+z^2,\ k=7$ is __**G**__.

Analyze: $7=x^2+z^2$.
Its graph is a right circular cylinder in *xyz*-space. Its axis is the *y*-axis.
Imagine taking a circle in the *xz*-plane and sweeping it parallel to the *y*-axis.

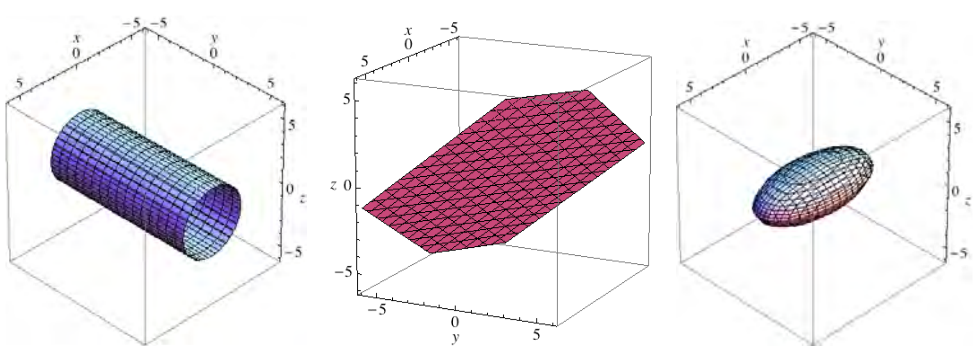
b) The level surface of $f\left(x,y,z\right)=3x+4y-5z,\ k=0$ is __**H**__.

Analyze: $0=3x+4y-5z$.
Its graph is a plane in *xyz*-space that passes through the origin.

c) The level surface of $f\left(x,y,z\right)=x^2+4y^2+9z^2,\ k=25$ is __**A**__.

Analyze: $25=x^2+4y^2+9z^2$.
Its graph is an ellipsoid centered at the origin in *xyz*-space.

Below are graphs of a), b), and c), respectively. (Courtesy *Mathematica*.)



10) Show that $\lim_{(x,y)\to(0,0)} \frac{3x^2-4y^2}{2x^2+3y^2}$ does not exist. (8 points)

Let $(x,y)\to(0,0)$ along the y-axis ($x=0$):

$$\lim_{(x,y)\to(0,0)} \frac{3x^2-4y^2}{2x^2+3y^2} = \lim_{(x,y)\to(0,0)} \frac{3(0)^2-4y^2}{2(0)^2+3y^2}$$

$$= \lim_{y\to 0} \frac{-4y^2}{3y^2}$$

$$= -\frac{4}{3}$$

Let $(x,y)\to(0,0)$ along the x-axis ($y=0$):

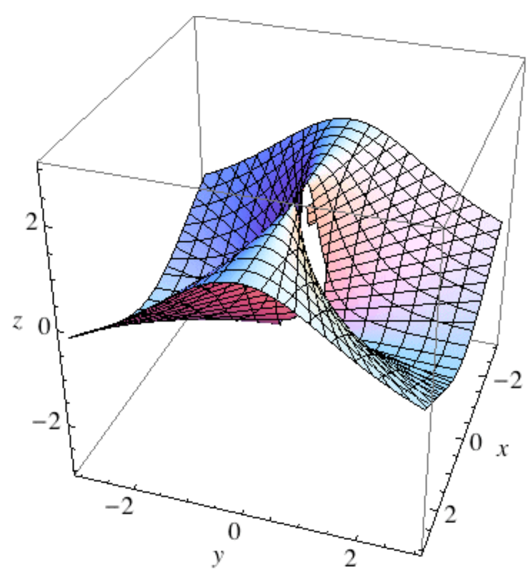
$$\lim_{(x,y)\to(0,0)} \frac{3x^2-4y^2}{2x^2+3y^2} = \lim_{(x,y)\to(0,0)} \frac{3x^2-4(0)^2}{2x^2+3(0)^2}$$

$$= \lim_{x\to 0} \frac{3x^2}{2x^2}$$

$$= \frac{3}{2}$$

We have found two paths approaching $(0,0)$ that yield different limit values for $\frac{3x^2-4y^2}{2x^2+3y^2}$, so the indicated limit does not exist by the Two-Path Rule.

Here is a graph of $z=\frac{3x^2-4y^2}{2x^2+3y^2}$:



11) Use polar coordinates to find $\lim_{(x,y)\to(0,0)} \frac{x^3y^3}{x^2+y^2}$. (7 points)

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{x^3y^3}{x^2+y^2} &= \lim_{r\to0} \frac{(r\cos\theta)^3(r\sin\theta)^3}{r^2} \\ &= \lim_{r\to0} \frac{(r^3\cos^3\theta)(r^3\sin^3\theta)}{r^2} \\ &= \lim_{r\to0} \frac{r^6\cos^3\theta\sin^3\theta}{r^2} \\ &= \lim_{r\to0} \left(r^4\cos^3\theta\sin^3\theta\right) \\ &= \boxed{0}\end{aligned}$$

Justifying the last step:

For all real θ, $-1 \le \cos\theta \le 1$ and $-1 \le \sin\theta \le 1$.

As a result, $-1 \le \cos^3\theta \le 1$ and $-1 \le \sin^3\theta \le 1$.

Therefore, $-1 \le \cos^3\theta\sin^3\theta \le 1$.

Observe that $r^4 > 0$ along any path approaching $(0,0)$ that avoids $(0,0)$, itself. Multiply all three parts by r^4 and apply the Sandwich / Squeeze Theorem:

$$\text{As } r\to0,\ \underbrace{-r^4}_{\to0} \le \underbrace{r^4\cos^3\theta\sin^3\theta}_{\text{So, }\to0} \le \underbrace{r^4}_{\to0}.$$

Informally, the limit of something approaching 0 times something that is bounded is 0.

Here is a graph of $z = \frac{x^3y^3}{x^2+y^2}$: