QUIZ 3 (SECTIONS 16.3-16.9) SOLUTIONS

MATH 252 – FALL 2007 – KUNIYUKI

SCORED OUT OF 125 POINTS ⇒ MULTIPLIED BY 0.84 ⇒ 105% POSSIBLE

1) Let $f(x,y,z)=\sqrt{3x^2y+z^3}$. Find $f_x(x,y,z)$. (4 points)

$$\begin{aligned}
f_x(x,y,z) &= D_x\left(\sqrt{3x^2y+z^3}\right) \\
&= D_x\left[\left(3x^2y+z^3\right)^{1/2}\right] \\
&= \frac{1}{2}\left(3x^2y+z^3\right)^{-1/2}\cdot D_x\left(3x^2\underbrace{y}_{\text{"\#"}}+\underbrace{z^3}_{\text{"\#"}}\right) \\
&= \frac{\cancel{1}}{\cancel{2}}\left(3x^2y+z^3\right)^{-1/2}\cdot\left[3y\left(\cancel{2}x\right)\right] \\
&= \boxed{\frac{3xy}{\sqrt{3x^2y+z^3}}}
\end{aligned}$$

2) Let $f(r,s)=\cos(rs)$. Find $f_r(r,s)$ and use that to find $f_{rs}(r,s)$. (6 points)

$$\begin{aligned}
f_r(r,s) &= D_r\left[\cos(rs)\right] \\
&= \left[-\sin(rs)\right]\left[D_r\left(r\underbrace{s}_{\text{"\#"}}\right)\right] \\
&= \left[-\sin(rs)\right]\left[s\right] \\
&= -s\sin(rs)
\end{aligned}$$

$$f_{rs}(r,s) = D_s\left[-s\sin(rs)\right]$$

We will use a Product Rule for Differentiation.

$$\begin{aligned}
&= \left[D_s(-s)\right]\left[\sin(rs)\right] + \left[-s\right]\left(D_s\left[\sin(rs)\right]\right) \\
&= \left[-1\right]\left[\sin(rs)\right] + \left[-s\right]\left[\cos(rs)\right]\left[D_s\left(\underbrace{r}_{\text{"\#"}}s\right)\right] \\
&= \left[-1\right]\left[\sin(rs)\right] + \left[-s\right]\left[\cos(rs)\right]\left[r\right] \\
&= \boxed{-\sin(rs)-rs\cos(rs)}
\end{aligned}$$

3) Assume that f is a function of x and y. Write the limit definition of $f_y(x,y)$ using the notation from class. (4 points)

$$\boxed{f_y(x,y)=\lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h}}$$

4) Let $f(x,y)=3xy^2-4y^3+5$. Use differentials to linearly approximate the change in f if (x,y) changes from $(4,-2)$ to $(3.98,-1.96)$. (12 points)

$$f_x(x,y)=D_x\left(3x\underbrace{y^2}_{\text{"\#"}}-4\underbrace{y^3}_{\text{"\#"}}+5\right)$$
$$=3y^2$$
$$f_x(4,-2)=3(-2)^2$$
$$=12$$

$$dx=\text{new } x-\text{old } x$$
$$=3.98-4$$
$$=-0.02$$

$$f_y(x,y)=D_y\left(3\underbrace{x}_{\text{"\#"}}y^2-4y^3+5\right)$$
$$=3x(2y)-12y^2$$
$$=6xy-12y^2$$
$$f_y(4,-2)=6(4)(-2)-12(-2)^2$$
$$=-96$$

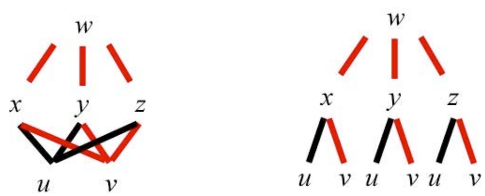
$$dy=\text{new } y-\text{old } y$$
$$=-1.96-(-2)$$
$$=0.04$$

The approximate change in f is given by:

$$df=\left[f_x(4,-2)\right]dx\ +\ \left[f_y(4,-2)\right]dy$$
$$=[12](-0.02)\ +\ [-96](0.04)$$
$$=\ \boxed{-4.08}$$

Note: Actual change ≈ -4.01315

5) Let f, f_1, f_2 and f_3 be differentiable functions such that $w = f(x, y, z)$, $x = f_1(u, v)$, $y = f_2(u, v)$, and $z = f_3(u, v)$. Use the Chain Rule to write an expression for $\frac{\partial w}{\partial v}$. (5 points)



$$\boxed{\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v}}$$

6) Find $\frac{\partial z}{\partial x}$ if $z = f(x, y)$ is a differentiable function described implicitly by the equation $e^{xyz} = xz^4$. Use the Calculus III formula given in class. Simplify. (9 points)

First, isolate 0 on one side: $\underbrace{e^{xyz} - xz^4}_{\text{Let this be } F(x,y,z)} = 0$

Find $\frac{\partial z}{\partial x}$. When using the formula, treat x, y, and z as independent variables.

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$$

$$= -\frac{D_x\left[e^{xyz} - x\overbrace{z^4}^{\text{"\#"}}\right]}{D_z\left[e^{xyz} - \overbrace{x}^{\text{"\#"}}z^4\right]}$$

$$= -\frac{e^{xyz}\cdot D_x\left(x\overbrace{yz}^{\text{"\#"}}\right) - z^4}{e^{xyz}\cdot D_z\left(\underbrace{xy}_{\text{"\#"}}z\right) - x\left(4z^3\right)}$$

$$= -\frac{e^{xyz}\cdot(yz) - z^4}{e^{xyz}\cdot(xy) - 4xz^3}$$

$$= \boxed{-\frac{yze^{xyz} - z^4}{xye^{xyz} - 4xz^3} \text{ or } \frac{z^4 - yze^{xyz}}{xye^{xyz} - 4xz^3} \text{ or } \frac{yze^{xyz} - z^4}{4xz^3 - xye^{xyz}}}$$

7) The temperature at any point (x, y) in the xy-plane is given by $f(x, y) = 2xy + x^2$ in degrees Fahrenheit. Assume x and y are measured in meters. Give units in your answers. (23 points total)

a) Find the **maximum** rate of change of temperature at the point $(3, 4)$. Approximate your final answer to three significant digits.

$$\begin{aligned}\nabla f(x, y) &= \left\langle f_x(x, y),\ f_y(x, y)\right\rangle \\ &= \left\langle D_x\left(2x\underbrace{y}_{\text{"\#"}} + x^2\right),\ D_y\left(2\underbrace{x}_{\text{"\#"}}y + \underbrace{x^2}_{\text{"\#"}}\right)\right\rangle \\ &= \left\langle 2y + 2x,\ 2x\right\rangle\end{aligned}$$

$$\begin{aligned}\nabla f(3, 4) &= \left\langle 2(4) + 2(3),\ 2(3)\right\rangle \\ &= \left\langle 14, 6\right\rangle\end{aligned}$$

The length of the gradient of f at $(3, 4)$ gives the **maximum** rate of change of temperature at that point.

$$\begin{aligned}\left\|\nabla f(3, 4)\right\| &= \left\|\left\langle 14, 6\right\rangle\right\| \\ &= 2\left\|\left\langle 7, 3\right\rangle\right\| \\ &= 2\sqrt{(7)^2 + (3)^2} \\ &= 2\sqrt{58} \\ &\approx 15.2\end{aligned}$$

Answer: About $15.2\ \dfrac{^\circ\text{F}}{\text{m}}$

b) Find the rate of change of temperature at $(3, 4)$ in the direction of $\mathbf{i} - 3\mathbf{j}$. Approximate your final answer to three significant digits.

Let **a** be the given direction vector $\mathbf{i} - 3\mathbf{j}$, or $\langle 1, -3 \rangle$.

Find the unit vector **u** in the direction of **a**.

$$\begin{aligned}
\mathbf{u} &= \frac{\mathbf{a}}{\|\mathbf{a}\|} \\
&= \frac{\langle 1, -3 \rangle}{\|\langle 1, -3 \rangle\|} \\
&= \frac{\langle 1, -3 \rangle}{\sqrt{(1)^2 + (-3)^2}} \\
&= \frac{\langle 1, -3 \rangle}{\sqrt{10}} \quad \text{or} \quad \frac{1}{\sqrt{10}}\langle 1, -3 \rangle
\end{aligned}$$

The directional derivative at $(3, 4)$ in the direction of **u** is:

$$\begin{aligned}
D_{\mathbf{u}} f(3, 4) &= \nabla f(3, 4) \bullet \mathbf{u} \\
&= \langle 14, 6 \rangle \bullet \left(\frac{1}{\sqrt{10}} \langle 1, -3 \rangle \right) \\
&= \frac{1}{\sqrt{10}} \left(\langle 14, 6 \rangle \bullet \langle 1, -3 \rangle \right) \\
&= \frac{1}{\sqrt{10}} (-4) \\
&= \frac{-4}{\sqrt{10}} \quad \text{or} \quad -\frac{4\sqrt{10}}{10} \\
&= -\frac{2\sqrt{10}}{5} \\
&\approx -1.26
\end{aligned}$$

Answer: $\boxed{\text{About } -1.26 \, \frac{^\circ\text{F}}{\text{m}}}$

c) Find a non-**0** direction vector **v** such that the rate of change of temperature at $(3, 4)$ in the direction of **v** is 0 [units].

We want a tangent vector to the level curve of f through the point $(3, 4)$.

Any non-**0** vector orthogonal to $\nabla f(3, 4)$, which is $\langle 14, 6 \rangle$, will do.

Observe that $\langle 6, -14 \rangle \perp \langle 14, 6 \rangle$, since $\langle 6, -14 \rangle \bullet \langle 14, 6 \rangle = 0$.

Therefore, $\langle 6\ [\text{m}], -14\ [\text{m}] \rangle$ or any non-**0** scalar multiple will do.

In particular, the simpler vector $\langle 3\ [\text{m}], -7\ [\text{m}] \rangle$ will do.

8) Find an equation for the tangent plane to the graph of the equation $5x^2 - 4y^2 + z^2 = 45$ at the point $P(-3, 2, 4)$. (10 points)

Observe that the given graph is a hyperboloid of one sheet.
(You may check that the coordinates of P satisfy the given equation, meaning that P lies on the graph of the equation.)

Isolate 0 on one side of the given equation.

$$\underbrace{5x^2 - 4y^2 + z^2 - 45}_{=F(x,y,z)} = 0$$

A normal vector for the desired tangent plane is given by $\nabla F(-3, 2, 4)$.

$$\begin{aligned} \nabla F\big|_P &= \left\langle F_x\big|_P,\ F_y\big|_P,\ F_z\big|_P \right\rangle \\ &= \langle 10x,\ -8y,\ 2z \rangle \end{aligned}$$

$$\begin{aligned} \nabla F(-3, 2, 4) &= \langle 10(-3),\ -8(2),\ 2(4) \rangle \\ &= \langle -30, -16, 8 \rangle \end{aligned}$$

An equation for the tangent plane is given by:

$$\left(F_x\big|_P\right)(x - x_0) + \left(F_y\big|_P\right)(y - y_0) + \left(F_z\big|_P\right)(z - z_0) = 0$$
$$(-30)(x - (-3)) + (-16)(y - 2) + (8)(z - 4) = 0$$

$$-30(x+3) - 16(y-2) + 8(z-4) = 0$$
$$\text{or} \quad 15(x+3) + 8(y-2) - 4(z-4) = 0$$
$$\text{or} \quad -30x - 16y + 8z - 90 = 0$$
$$\text{or} \quad 15x + 8y - 4z + 45 = 0$$